THE MATHEMATICAL GAZETTE

EDITED BY
T. A. A. BROADBENT, M.A.

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VOL. XXI. MAY, 1937. No. 243

FRANCIS SOWERBY MACAULAY

11 February 1862—9 February 1937.

SCHOOL MATHEMATICS AT THE BRITISH ASSOCIATION.

THE British Association is holding its Annual Meeting this year at Nottingham from 1st September to 8th September. The mathematics subsection is having one symposium on the teaching of geometry in schools and a second one on the teaching of algebra in schools.

Further particulars may be obtained from Dr. D. M. Wrinch, Lady Margaret Hall, Oxford, who will be glad to hear of anyone wishing to contribute to either symposium in the general discussions.

THE DEVELOPMENT OF THE TEACHING OF GEOMETRY IN GERMANY.

By Georg Wolff, Düsseldorf (Germany).

[At the evening meeting of the Annual Meeting of the Mathematical Association on 5th January, 1937, Mr. A. W. Siddons took the chair, and in calling upon Professor Wolff for his paper said: Professor Georg Wolff came to England in 1913, and I believe I am right in saying that Harrow was the first school to which he came and that the last educational institute he was at in England was this Institute of Education in Southampton Row, where he talked with Sir Percy Nunn. When he was in England on that occasion Professor Wolff spent much of his time learning about the teaching of mathematics in this country. He has more recently been lecturing at Columbia University, New York.]

1. *The past and present teaching of geometry in England.*

If we compare in general the development of the teaching of geometry in England with that in Germany, we find that they are similar in so far that, in both countries, in the beginning the correct teaching of the textbook of geometry was considered to be a translation of the language of Books I-VI, XI and XII of *Euclid's Elements*. The historical growth of the teaching of geometry in these two countries is similar in so far as at some time there became apparent a movement against the use of Euclid as a textbook for children who, not being able at the time to think in such terms, needed to learn how to think. And thus in both countries a major portion of the geometry course is separated from Euclid and is made palatable for children. In these two countries, the present problem is: "Is there still any need for the geometry of the Greeks?"

The developments of the teaching of geometry in these two countries differ in detail from one another. In order to understand this difference, it is necessary to trace the main epochs of the geometric education in these two countries. My friend, Mr. A. W. Siddons * of Harrow gave an excellent sketch of this history in his Presidential Address to the Mathematical Association in January, 1936, and I myself have been interested in this question in surveying the mathematical education in England.†

Regardless of the fact that the First and Second Royal Commissions of 1864 and 1868 rejected Euclid as a school textbook, and in spite of the fact that the Association for the Improvement of Geometrical Teaching (A.I.G.T.) that was formed in 1870, created important courses of study, it was necessary to wait until the beginning of this century for the *revolt against Euclid* to make definite progress.

The most destructive attacks were conducted: 1. By this Mathematical Association, which was organised in 1897, and its official journal, *The Mathematical Gazette*. 2. By Professor John Perry

* Siddons, A. W., "Progress", Presidential Address to the Mathematical Association, January, 1936. *The Mathematical Gazette*, No. 237, 1936, p. 7.

† Wolff, Georg, *Der Mathematische Unterricht der höheren Knabenschulen in England*. B. G. Teubner, Leipzig, 1915.

during the meetings of the British Association for the Advancement of Science in Glasgow in 1901, and in Belfast in 1902 ; and finally, 3. By Professor Forsyth, who to my great pleasure is presiding at this Association.

In 1903, the geometry problems of Entrance Examinations in Oxford and Cambridge were finally freed from the dominance of Euclid. This was a triumph and accomplishment of great importance and the school mathematicians thus attained the aims they had been striving for for a long time.

Among the stream of the new textbooks, one could see that many authors took a conservative position during the transition period and could not allow themselves to make a change in a hurry, while others went to the extreme, and, again, a third group followed a practical middle path. In this last group I would place the books of W. C. Fletcher,* formerly of the Board of Education, and those by Godfrey and Siddons. †

A new impetus to geometrical thought was given by the International Commission on the Teaching of Mathematics founded in Rome in 1908 which demanded of geometry the strengthening of space perception and the experimental practical application. We know that England took a leading position on this commission because Professor George Greenhill was Vice-President of the International Commission. We also know that the British Sub-commission under the authority of the Board of Education in its very important and excellent reports on the Teaching of Mathematics in the United Kingdom, published in 1912,‡ contributed towards the world-reform of mathematical education. And, finally, we know that the most successful meeting of the Commission took place in this country in 1912, at which was read the prominent report of Professor David Eugene Smith of New York on " Intuition and Experiment " since then proved to be influential for many years.

The crowning of this phase of development is associated with the famous publication of the Board of Education of March, 1909, Circular 711, on the *Teaching of Geometry and Graphical Algebra*, by W. C. Fletcher. In this publication we can note the great change that then took place in geometry teaching in England. In it we recognise the complete break with the dominance of Euclid and the tying up of the geometric structure with interest, the interpretative values, and the understanding of the mind of a child. This was a new orientation, and in no other country was it more revolutionary. It bears witness how greatly influential were the ideas of the International Committee on the Teaching of Mathematics in this country.

This circular demanded that the quintessence of geometric education be in the method of teaching. This method should consist of three stages.

* Fletcher, W. C., *Elementary Geometry*. London, Edward Arnold.

† Godfrey, C., and Siddons, A. W., *Elementary Geometry*. Cambridge, 1903.

‡ Board of Education, Special Reports on Educational Subjects, Vols. 26 and 27. *The Teaching of Mathematics in the United Kingdom*, Parts I and II. London, 1912.

In the *first stage*, the pupil should be introduced through observation and experiment and a minimum of explanations into the language and the concepts of geometry. No mention of axioms and postulates should be made.

Then follows the *second stage* of the discovery of the properties of figures. The pupil learns certain propositions by means of drawing, measuring, paper-folding, and other investigations. Construction problems should be avoided. Parallels should be introduced through the notion of direction.*

Finally, in the *third stage*, construction problems and propositions should be worked on—deductive, not as mental acrobatics, but by means of a good mixture of observation and thinking.

As in all other lands, not all the mathematicians of England joined the new camp whole-heartedly. Even in 1913 I observed an English schoolmaster teach geometry from Todhunter's Euclid, and in the same year a prominent English philosopher and mathematician told me, "It is a pity that we never have Euclid in English schools".

However, the pendulum of the reform was not standing still. The numerous Reports of the Mathematical Association, which appeared before and after the War, show the activity on behalf of the improvement of the teaching and methods in this country.

Among these Reports, the outstanding is "The Teaching of Geometry in Schools" of 1923. Historically it is tied up with the above-mentioned circular, and it follows it up with a *fourth stage* with modern geometry, geometrical conics, and solid geometry, and finally a *fifth stage* devoted to foundations of geometry.

In the book by A. W. Siddons and R. T. Hughes, Part One, *Practical Geometry*, and Part Two, *Theoretical Geometry*,† many recommendations of this excellent Report are put into practice.

Unfortunately, I am not familiar with the ideas of the Teaching Committee which it will propose in its next Report shortly to be made public. Therefore, I must conclude my short review of geometric education in England and proceed to the history of geometric education in Germany.

2. *The teaching of geometry in Germany prior to the Merano reform* (1905).

Since the Middle Ages, mathematics was taught in the monastery and Latin grammar schools. But this was mainly confined to arithmetic and algebra and very little geometry, perhaps up to the theorem of Pythagoras.

Later the mathematics situation was improved; in 1810 the State took over the supervision of the schools, and in 1816 a State course of study was published which prescribed six hours weekly to mathematics for every class. Although already Gottfried Wilhelm

* Garstang, T. J., *Parallel Straight Lines and the Method of Direction*. Report of the International Commission, Vol. I, p. 274.

† Cambridge University Press.

Leibniz (1646-1716), the great contemporary of Sir Isaac Newton (1642-1727), expressed views against the use of the *Elements of Euclid* as a textbook in geometry,* they were conservative in this strict system.

And it did not take long until school textbooks that adhered strongly to the system of Euclid, but at the same time contained material not so bare, naked, and impersonal, but which tried to bring in a live, warm and a personal note, were published.

Moreover, other authors attempted to present the series of propositions according to the Greek pattern but in a strictly logical and at the same time shorter form, so that applications might be introduced. Besides, some of the teachers introduced conic sections in the upper classes and also attempted to teach projective geometry.

When we examine the development of geometrical education in the nineteenth century, we note the following movements which are more or less of an influence on the changes in teaching.

(*a*) *Pedagogy.* There is no doubt that the European pedagogy of the nineteenth century was dominated by the ideas of Johann Heinrich Pestalozzi (1746-1827), Johann Friedrich Herbart (1776-1841), and Friedrich Fröbel (1782-1852). Their aims in education can be summarised as follows : Through empirical observation to thinking, learning through concrete facts, independent work, the use by children of the play-element in the process of learning, the child should be led to creative activity.

It was often mentioned by mathematicians that Pestalozzi † and Herbart ‡ and their close circle promoted directly the science of method of mathematical education by their writings. It must, however, be said that this was only to a slight extent the case. First such men as Adolph Diesterweg (1790-1866),§ Harnisch (1787-1864),‖ and the famous Georg Simon Ohm (1789-1854)¶ and many others who had mathematical training and who had realistic aims in the new formulation of geometry, were responsible for great strides forward. The revolt against Euclid, against rigidity and deadening formalism, against abstraction and exaggerated pedantry was begun. The new school books were live and enticing to pupils, the figures were far from monotonous, they were artistically arranged in various beautiful forms, the mathematical presentations were illustrated by actual objects.

But the system of Euclid remained. The outer form and framework, the inside and outside were presented in a new shape.

*Gerhardt, C. J., Leibniz, G. W., *Mathematische Schriften*, Bd. 7, p. 141 ff. Halle a. d. Saale, 1849-63.

† *ABC der Anschauung oder Anschauungslehre der Massverhältnisse.*

‡ Herbart placed the triangle in the foreground.

§ Diesterweg, A., *Leitfaden für den ersten Unterricht in der Formen—Grössen und räumlichen Verbindungslehre.* Elberfeld, 1822.

‖ Harnisch, W., *Die Raumlehre oder die Messkunst, gewöhnlich Geometrie genannt.* Berlin, 1821.

¶ Ohm, G. S., *Grundlinien zu einer zweckmässigen Behandlung der Geometrie.* Erlangen, 1817.

We must not forget that to Pestalozzi and Herbart we owe one part of the creation of the Realschule, later on Oberrealschule or Modern Side of a Grammar School in England with its practical understanding of the needs of the present day. In the Realschule, besides modern languages, mathematics and natural sciences were taught. Those schools represented a new movement in the middle of the century. In 1882 they received the first State recognition, and in 1900 they were fully appreciated, and the most recent reform, which we await now, will perhaps push the realistic education still farther forward and very few of the old Grammar Schools will be left.

With this new orientation there began an enlargement of the geometry subject matter which was introduced in the curriculum besides *analytic geometry* also *synthetic geometry, i.e.* the study of *conic sections* with Greek mathematics.

(*b*) *Descriptive geometry.* We always tried to introduce into the German class-room the influence of the development of our science. And thus there was a group of school teachers who made experiments in order to find out the reactions of the pupils to *descriptive geometry.*

In contrast to Euclidean geometry, this developed from the most general transformation, from the *perspective.* Only about four hundred years later there followed the special cases of *orthogonal projection* and *oblique projection.* In contrast to Euclidean geometry, it did not require strong logical thinking for its formulation. It developed on account of the immediate needs of solving practical life problems of such Italian artists such as Philippo Brunelleschi * (1377-1446), Leon Battista Alberti (1404-1472), Donatello (1386-1468), Lorenzo Ghiberti (1381-1455). Their main problem was, " How can we reproduce objects in accordance with the process of seeing ? "

We know that until the sixteenth century, painters and architects grappled with this problem until Simon Stevin (1548-1620) in the north and Guido Ubaldo del Monte (1545-1607) in Italy attempted the first synthesis.

But the inner development of *perspective* continued for two more centuries.

It is my duty here to point out the contributions of England towards the maturity of this science. At a time when perspective was only being considered speculatively, the Franciscan Roger Bacon (1214-1294)† was already preoccupied with it, and so was John Peckham (1240-1292), the famous Archbishop of Canterbury.‡ In 1602 Ditton's *Perspective* was published and afterwards there followed the epoch-making works of Brook Taylor (1685-1731), 1. *Linear Perspective*, 1715, and 2. *New Principles of Linear Perspective*

* Wolff, Georg, *Mathematik und Malerei*, 2. Aufl. B. G. Teubner, Leipzig : p. 49.

† *Perspectiva.* Frankfurt, 1614.

‡ *Perspectiva Communis*, 1 ed. Mailand, 1492, later appeared in Nürnberg in 1542.

in 1719, which caused a stream of new English works on this discipline. Joseph Priestley * considered perspective from the historical and physical points of view.

Only afterwards was central projection completely developed, and in the eighteenth century foundation was given to oblique projection, as, for example, by Johann Heinrich Lambert (1728-1777) in *Freye Perspective*, Zürich, 1759.

And not till then followed Gaspard Monge (1746-1818) with his descriptive *Geometry* (1789-99), completing by means of his orthogonal projection the system of descriptive geometry or system of projection.

(*c*) *Projective geometry.* By means of this new kind of thinking in geometry there developed a new independent geometry, known to us as Projective Geometry. Its origin is associated with men such as: Girard Desargues (1593-1662), Jean Victor Poncelet (1789-1867), Jacob Steiner (1796-1863), Christian von Staudt (1798-1868), George Walker (1735-1807), Arthur Cayley (1821-1895), and others.

It suffices to note its characteristics: I. It is based on a few fundamental concepts, quite naturally and with no artificialities.

II. It does not contain the parallel axiom or any philosophical difficulties.

III. In the foreground is the *principle of projection* and of the generation of new configurations. One conceives the conic sections by means of pencils, by means of ranges, and the relation between ranges and pencils.

IV. In it the pupil is working mostly by the use of *inductive conclusions*; he uses analytic methods for the discovery of new properties. Thus the properties of the circles are tied up with conic sections by means of projections.

But the *deductive method* is not totally absent there. In the study of conic sections are recapitulated: the point, straight line, circle, ellipse, parabola, hyperbola all considered as a whole, as one configuration depending on the angle under which the cutting plane is drawn. The ellipse is determined by five points or five tangents, and not by means of two foci and one other point or by similar combinations due to the Greeks.

V. In order to discover new properties of figures we employ the *principle of motion* which is also utilised for the determination of positional relations. Consequently in the order of the day we have displacement, rotation, reflexion, contraction, expansion, the section of figures and solids, the transition from secants to tangents, the transition from a hexagon to a pentagon, to a quadrilateral, to a triangle, the interrelation between the various conic sections, the parabola and hyperbola as special cases of an ellipse, the degeneration of conic sections, and so on.

* Priestley-Klügel, *Geschichte u. gegenwärtiger Zustand der Optik.* Leipzig, 1775. J. Priestley, *A familiar introduction to the Theory and Practice of Perspective.* London, 1770.

VI. Then there is also the *principle of transformation*. Propositions which refer to points are restated to embrace straight lines and finally planes. We refer to the principle of duality, and to the principle of reciprocity. For example, pole-polar, complete quadrilateral—complete quadrangle, Pascal—Brianchon, equilateral triangle—tetrahedron, generation of curves by points and accordingly by straight lines, locus-envelope—a point on a straight line—straight lines passing through a point, range—pencil, finally also sheaves of planes, pole and polar of a circle and a sphere.

VII. Thus we see that new ideas are being applied to geometry, ideas unknown to and not utilised by Euclid. We observe that *dependences, variation, life and stimulation* are introduced into the thinking process as the student of this geometry is enticed to traverse the path of inquiry. In this geometry there is no definite plan of proof except that every proof is many-sided.

VIII. It is also significant that it is necessary to do *manual work*, it is necessary to draw and measure. This activity is interesting to many, who thus find an easier way to make discoveries. The joy of this no doubt results in a love for geometry. Newton was quite right when he once said :

" The most important thing in geometry is that one finds the way to discovery."

(*d*) *Applied mathematics.* But school geometry has still many other tasks to fulfil. We should recall its applications to the extensive fields of the theory of measurement, mapping, astronomy, trigonometry, photogrammetry, machine-drawing and the drawing of various objects, in handwork and technology, in representations in physics.

We refer nowadays to these fields as those of applied mathematics, as the domains which assigned to mathematics a prominent place and meaning in the evolution of civilisation.* And it is quite certain that the new methods of drawing and the theory of projections were very helpful and useful tools in the discovery of the application.

(*e*) *The* 1901 *course of study.* Should we recall the requirements set by the great educators, the descriptive geometers, the proponents of projective geometry, and the applied mathematicians we would find common agreement among them, somehow as Oswald Spengler spoke of the parallelism of the world-rising in his *Decline of the West.*

For the same reason the demands of these men knocked at the doors of the schools, repeatedly demanding admission. This occurred frequently in Germany about and since 1820, with special energy about 1864. Some courageous teachers based their teaching on these new movements, they wrote treatises and books on their experiences and aims, even though the Ministry of Education did not adopt their suggestions in the courses of study. Thus the

* The best introduction to this very complex problem may be found in Ernst Tiedge, *Bildungsaufgaben des mathematisch naturwissenschäftlichen Unterrichts der höheren Schulen.* M. Diesterweg, Frankfurt, a.M., 1933.

courses of study of 1882 and 1892 neglected the new ideas. We had to wait until 1901.

This course of study followed the demands of the educators and the adherents of projective geometry through the following measures:

I. The study of geometry, contrary to previous practice, need not begin with definitions. It should be preceded by an intuitive stage to which one hour per year or two half-hours per year should be devoted in the Oberrealschule.

II. The number of propositions should be reduced. Above all, teachers should attempt to make more use of construction problems, and by means of their discussion to discover new geometric properties. This also signifies a definite break with the traditional Greek procedure : statement, generalisation, proof, and the introduction of the method of projective thinking.

III. In the upper grades new material as " modern geometry " was studied. Ceva's and Menelaus' theorems, harmonic ranges and pencils, complete quadrangle and complete quadrilateral, pole and polar, the radical axis of circles, inversion, Apollonius' problem of contact.

IV. In the Realanstalten (schools in which French, English, Mathematics, and Science were taught), a little introduction to descriptive geometry, based mainly on the method of Monge, was provided.

V. Besides, two hours per week were set aside for drawings in descriptive geometry, called Linearzeichnen, for which the students enrolled voluntarily.

If we survey all these innovations we immediately recognise a step forward, which, however, did not satisfy all alike. We notice everywhere additions in order to satisfy the demands of the educators and the geometers, but these were only additions. The purpose was to plant the seeds of the *creative powers of the projective method* in the German geometry course, the aim was to give by means of Modern Geometry a view into the new material. But the methods which have been in use thus far followed predominantly the path of Euclid. They presented *no* distinct image of that which we aimed to call " the spirit of projective geometry ".

3. *The Merano course of study and its effects.*

The German Sub-committee of the International Commission on the Teaching of Mathematics, stimulated by the reform of 1902 in France, began its activities in 1903, at that time called the Committee of the German Association of Scientists and Doctors (Kommission der Gesellschaft deutscher Naturforscher und Ärzte). This committee was confronted with the above-described situation in the teaching of geometry.

We now know that the chief aim of the great international reform was the introduction of the function, the calculus and graphs, and that these ideas predominantly affected arithmetic, algebra and

analysis. The multitude of problems thus introduced enabled teachers above all to solve them satisfactorily. Owing to this fact, we should openly confess it, the importance of geometry in Germany was somewhat decreased. Less time and effort were allotted to its advancement.

At the meeting in Merano in 1905 the Commission proposed a course of study known now as the Merano Course of Study.* With your permission I will outline briefly its parts pertaining to geometry :

I. Configurations should always be considered as movable objects, and thereby it will be possible to examine their interdependence.

II. Whatever can be arrived at intuitively should not require any proof.

III. Geometry should not be confined to the plane only, but plane and solid geometry should be permanently interwoven.†

IV. The teaching of drawing in connection with mathematics was accentuated. Construction methods for the correct drawing of objects were emphatically stressed.

V. In order to foster empirical perception models were employed. Such models were also constructed.

VI. Applications of geometry were also to be taught as well as the history of geometry.

Some textbooks went somewhat farther than these recommendations. They introduced axial and central symmetry. Limits were little used except in the case of secants and tangents.

This course of study was in a strange situation. Its authors were men who surrounded Felix Klein, subsequently President of the International Commission on the Teaching of Mathematics, and it had no official governmental endorsement. Notwithstanding this, all school teachers were in favour of this course of study, new textbooks for classroom use were published ; briefly, the reform was already under way at a time when no official orders on the subject had been given.

4. *The quiet reform.*

At the beginning of the World War there arose a new movement. The train of thought was as follows :

The Merano course of study aimed at promoting those parts of mathematics that have to do with number, and thus by means of the function-concept arithmetic, algebra, analysis, and analytic geometry were unified in a single whole. This unification proved to be very useful and purposeful.

* Klein, F., *Vorträge über mathematischen Unterricht*, bearbeitet von Rud. Schimmack. B. G. Teubner, Leipzig, 1907 ; Schimmack, Rud., *Die Entwickelung der mathematischen Unterrichtsreform in Deutschland*. B. G. Teubner, Leipzig, 1911.

† Schimmack, when delivering his inaugural lecture (1911), laid historical foundation to this fusion. See *Zeitschrift für math. und naturwissenschaftlichen Unterricht*, Vol. 42, 1911, p. 569.

In contrast to this the position of geometry was very gloomy. We teach heterogeneous geometric disciplines—standing very near one to another, but apparently having no inner connection. Thus we have :

(*a*) The Euclidean geometry with a projective wrapping.
(*b*) Modern geometry—a mixture of Euclidean and projective approach.
(*c*) Descriptive geometry—as a parallel projection.
(*d*) Analytic geometry.

Is it possible, we ask, to blend all these geometries into one ? Is it not possible to find a connection between them ?

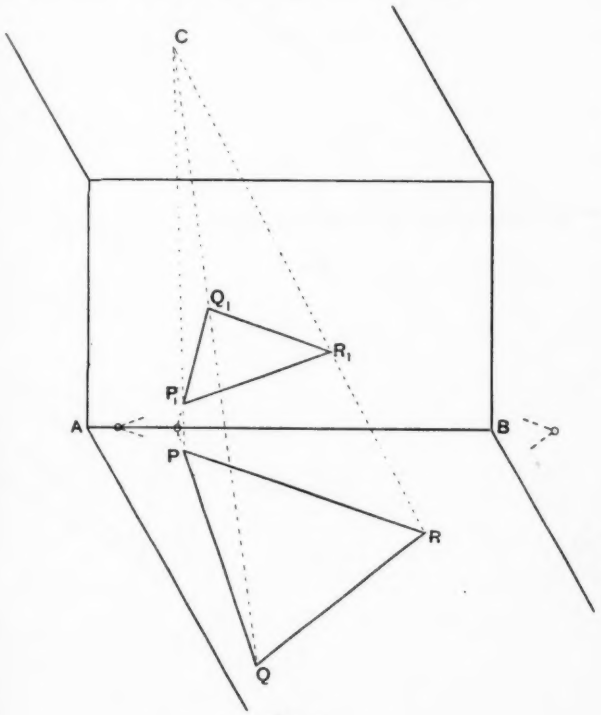
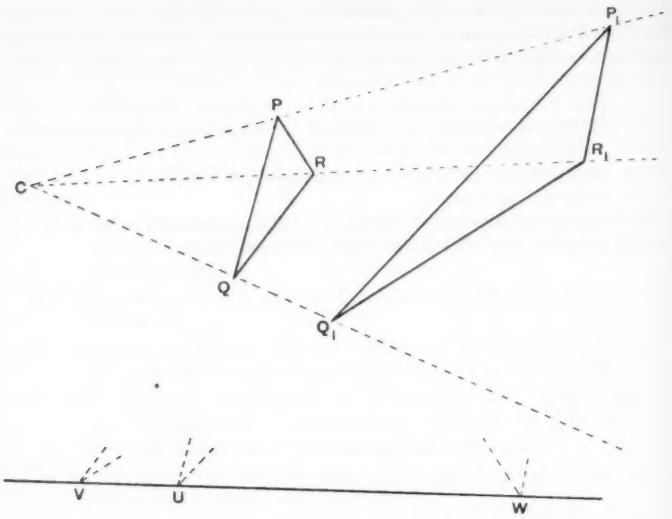


FIG. 1.

Already Leibniz (1646-1716) had been attracted by this problem while searching for a *principle of order in geometry*. He contrasted *analysis* with *geometry*, *constancy* (Stetigkeit) with *continuity* (Kontinuität), and thereby *function* with *depiction*. Leibniz named this parallelism *the principle of continuity*.



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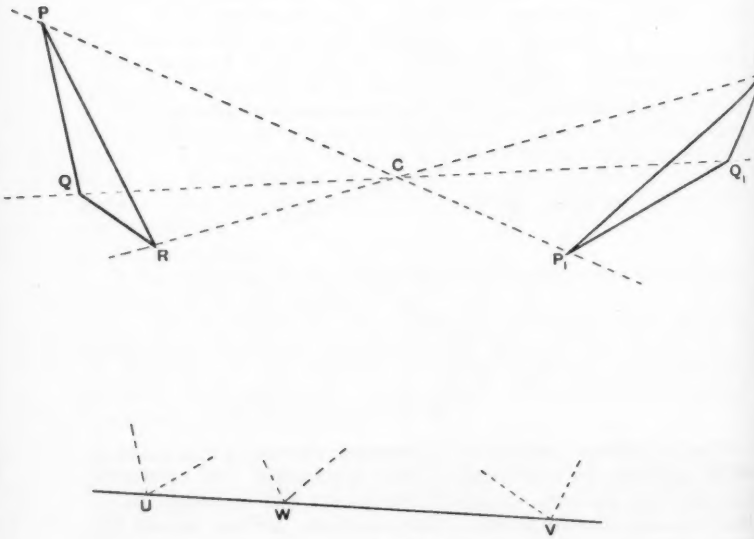

FIG. 3.

The next question was :

Could this idea of structure give a true inter-connection between the various geometries studied in schools ?

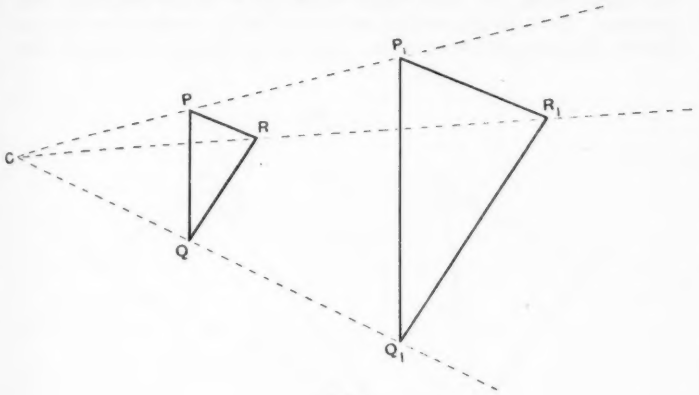

FIG. 4.

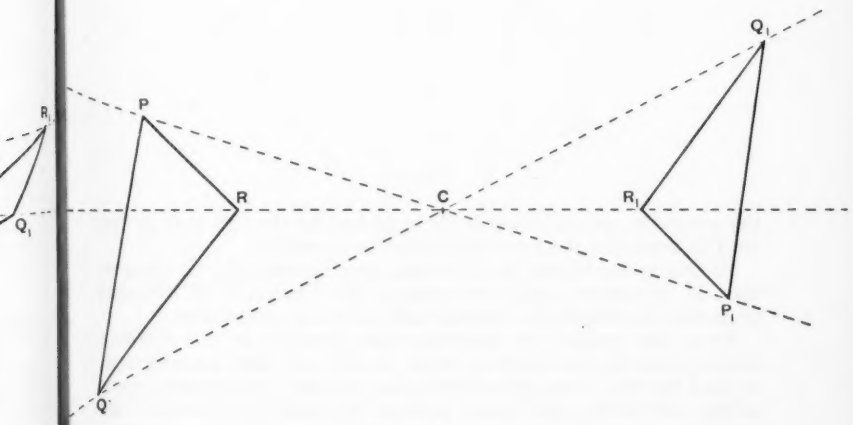

FIG. 5.

If we pursue the development of descriptive and projective geometry, we find that the well-known geometer Wilhelm Fiedler of Zürich (1832-1912) accomplished the unification of these two disciplines. The projection of one figure in one plane into another figure of another plane, depicted in perspective, involves a relation

between the original figure and its projection which is called *central collineation, centro-collinear relationship,* or *depiction in perspective.*

After this systematisation of the geometries there remained one last step, made by Felix Klein (1849-1925) in his Erlangen Programme. In it the geometries are subdivided in conformity with the theory of groups and invariants. And lastly, our geometries fall into

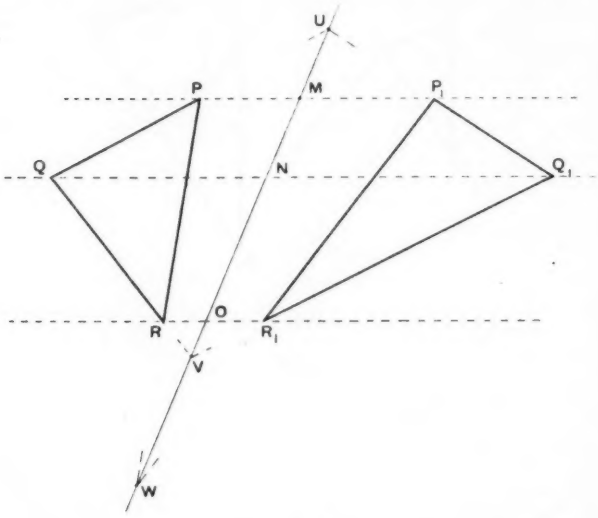

FIG. 6.

the projective group, which is distinguished by the fact that in this kind of depiction the cross-ratio remains invariant.

To this group belong the following transformations: I. Congruence, II. Symmetry, axial and central, III. Similarity, IV. Parallel projection or affinity, V. Central collineation or perspective.

From this outline we conclude that formerly in our German school-geometry, as inherited from Euclid and then transformed, we had the first three transformations, namely: congruence, symmetry, similarity, and some portion of parallel projection—the oblique projection. Orthogonal projection must therefore be considered from the point of view of a projective depiction and finally must yet find the perspective applications.

And this is exactly what happened. The perspective (Figs. 1, 2, 3) is the most general of all projective transformations, all others are special cases. If, for example, the picture plane is parallel to the object plane, we have *similarity* (Figs. 4 and 5); if the centre is at infinity, we have *parallel-projection* or *affinity* (Fig. 6); if the axis

of affinity is at infinity or if in parallel-projection the picture plane is parallel to the object plane, we have *congruence* (Fig. 7). Axial

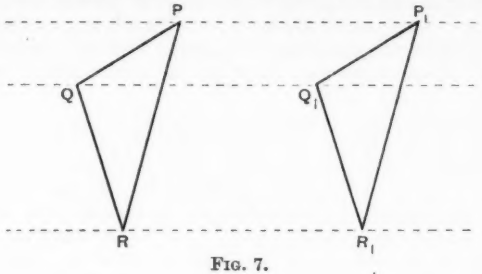

FIG. 7.

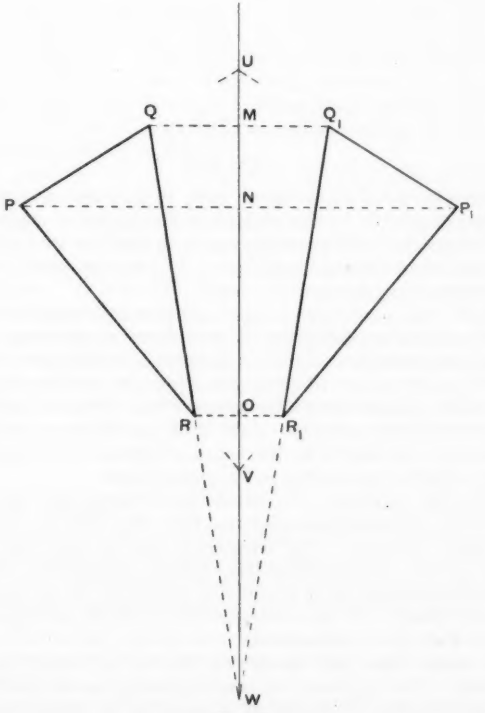

FIG. 8.

symmetry is originated from affinity, when the representation-relation is $q=1$ or when $QM=Q_1M$ (Fig. 8). And finally: central

symmetry is a special case of similarity (Fig. 5), if $CP = CP_1$ (Fig. 9).

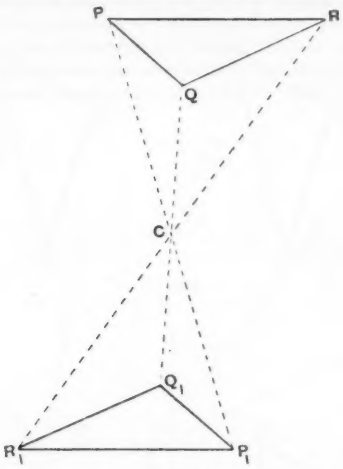

FIG. 9.

But the effects of these ideas were still more impressive; the recommendations of the International Commission concerning the *fusion of plane and solid geometry* result in the fact that this method of approach recognises no solid geometry, except that it is a space and plane geometry in one.

The work with projective ranges and pencils immediately reveals simplicity and far-reaching meaning in every application up to and including conic sections. Pascal's theorem is better proved by means of projection, by means of perspective pencils, than by means of the theorem of the ancient Greek Menelaus (about 100 A.D.) who followed Euclid's method thoroughly. And thus we did away with the so-called Modern Geometry in the sense of Euclid and restated it in terms of projective geometry in its purest form.

And here the geometry of position leads us through the *principle of dynamics* to infinitesimal analysis, thus the *limit*, so to speak, has a wider inner meaning. The transition from a secant to a tangent might be refined, and the transitions from a polygon to a circle, from a prism to a pyramid, from a prism to a cylinder, from a pyramid to a cone, and finally the calculation of the volume of a sphere might be treated with more refinement.

At the same time, this geometry directly indicated its possible applications. The emphasis on scale drawing necessitated the solution of construction problems as a problem in measurement. This was demanded especially in technology and other cases of practical life where correct representations by drawings were required. Another example of this was the converse problem of the perspective

—photogrammetry, and its numerous applications to nature, architecture and painting.

Thus we see how we, in Germany, teach *one* unified geometry*, how the movement *away from Euclid* succeeded beautifully. We further see how the development of projective and descriptive geometry and of applied mathematics, about which we spoke here, found one unified idea.

But our previous reference to education, descriptive geometry, projective geometry, practical geometry would not be complete if pedagogy had not found the correct procedure. The educator Georg Kerschensteiner (1854-1932) presented to us his ideas about the school as a workshop (Arbeitsschule).† According to these, practical cooperations of the pupil in drawing or other activity as a worker has a definite meaning for him. This is the laboratory method which is described in the Three Reports of Kerschensteiner of 1913, as well as in many articles. I observed an ideal realisation of this, twenty-five years ago, under the direction of my friend A. W. Siddons ‡ in Harrow. *This is the method of teaching mathematics as an empirical natural science, as applied mathematics.* Even if Kerschensteiner had never lived, the pedagogical setting of projective geometry would eventually and naturally have led to the activity or Arbeitschul-method of teaching.

We are pleased that the German course of study of 1925 § completely follows the ideas of this " quiet reform ". It rightfully demanded that at the completion of the study of geometry an insight into the system of Euclid as a perfect example of logical strength from both the philosophical as well as from the historical point of view be given. It also demanded the inclusion of a survey of the history of the parallel axiom.

It is also worth while to note the introductory stage of the study of geometry mentioned previously. The new development of school geometry begins immediately as the child enters school. It concerns itself with the study of the square, allows him to examine planes, lines, points, not halting to examine axioms, postulates and definitions.

Conclusion.

Finally, we inquire :

What is left of Euclid in the German school geometry ?

Allow me to answer this question briefly :

(*a*) The characteristics of the Euclidean geometry, pure geometry, a strong logical development of an abstract system, *a priori*.

* Reidt–Wolff–Kerst, *Die Elemente der Mathematik*. 4 Hefte Arithmetic, 4 Bände : Algebra, Analysis, Analytische Geometrie, Geometrie. G. Grote, Berlin, Dessauerste.

† Kerschensteiner, G., *Wesen u. Wert d. naturwissenschaftlichen Unterrichts*, 1. Aufl., 1913, 3. Aufl., 1928, B. G. Teubner, Leipzig.

‡ Siddons-Vassal, *Practical Measurements*. Cambridge University Press, 1912.

§ *Richtlinien für die Lehrpläne an den höheren Schulen in Preussen*, Weidmann, Berlin, 1925.

(*b*) What is the nature of Euclidean geometry in schools nowadays ?

1. The influence of group theory : transformations, projectivity, symmetry, parallel projection, similarity in terms of perspective motion, changes in figures, considerations of limits.

Were these ideas known to Euclid ?

2. The influence of applied mathematics and pedagogy. The *a posteriori* perception, physical point,* physical straight line, plane, solids ; methods of independent activity, methods of drawing, modelling problems, workshop problems, practical problems, laboratory exercises ; measurements of lengths, heights, surfaces.

Were these ideas also known to Euclid ?

Have we, therefore, a right to speak of Euclid in the schools ? Should we not find courage to make the last departure from Euclid—away from Euclid ? This is the question which must be answered in the near future.

Allow me to survey once more the path which we all traversed today. We have surveyed the teaching of geometry in England and Germany, and we should admit that in these two countries † above all we have the common tendency of development :

Away from Euclid. G. W.

GLEANINGS FROM FAR AND NEAR.

1109. " Vigilant " writes from North Wales :—A visit to a holiday resort at this time of the year affords unrivalled opportunities for studying the youth and manhood of our country in the possession of high-powered sports cars. Nowadays every scientific investigator insists on reducing the behaviour of the object studied to some kind of algebraic formula, and I therefore submit this :

x	$=$	h	$\times$	n	$\times$	i
a constant quantity		horse power of car		noise produced by same		intellectual capacity of owner

It follows from this equation that as the quantity " i " diminishes, often to the vanishing-point, a great strain is put upon the manufacturers of sports cars to increase the factors " h " and " n " so that " x " may remain constant. Fortunately by this time the degree of mental feebleness is so marked that the subjects could without injustice be isolated for special treatment, and the ample funds previously devoted to sports cars of ever-increasing complexity and noisiness made available for the fees of the institutions to which they are transferred.—*Manchester Guardian*, August 8, 1935. [Per Mr. A. P. Rollett.]

* Hjelmslev, H., " Die Geometrie der Wirklichkeit ", *Acta Mathematica*, Vol. 40, 1915, pp. 35-66 ; Hjelmslev, H., " Die natürliche Geometrie ", *Abhandlungen aus dem Math. Seminar der Hamburger Universität*, Vol. 2, 1933, p. 1, obtainable from B. G. Teubner, Leipzig.

† It may be mentioned that U.S.A. has the same tendency. The extremely excellent book of Smith-Reeve-Morss, *Text and Tests in Plane Geometry*, Ginn and Co., New York and London, is a very good example in this direction.

POPULATION TRENDS.*

By R. R. Kuczynski.

Population trends are determined by fertility and mortality. The crudest method of measuring fertility and mortality consists in relating the number of births and of deaths to the total population. The birth and death rates thus obtained show the proportion by which a population increases through births and decreases through deaths. But since these rates are calculated without regard to the sex and age composition of the population they do not afford an adequate gauge for the measurement of fertility and mortality. The disturbing influence of the sex and age composition is eliminated by computing fertility tables and life tables.

The concept of a life table is first to be found in the *Natural and Political Observations upon the Bills of Mortality*, published in 1662 by John Graunt. This remarkable man, a London haberdasher, was indeed not only the first to relate the number of deaths to the total population and thus to compute what we now call the crude death rate, but he tried also to show how many "of 100 quick conceptions . . . die before they be six years old", etc. His assumptions about the number of deaths at various ages, it is true, were mere guesswork and his argument suffered from several serious defects, but the life table which he constructed for London nevertheless was a monumental achievement inasmuch as it was the first attempt to compute the number of survivors at various ages on the basis of the deaths registered for a given territory.

Thirty years after the publication of Graunt's *Observations*, the Royal Society submitted Caspar Neumann's investigations of the mortality in Breslau to the great astronomer Edmund Halley, asking for his opinion. Halley secured from Neumann additional material and thereupon wrote his famous article "An Estimate of the Degrees of the Mortality of Mankind, drawn from curious Tables of the Births and Funerals at the City of Breslaw".† He formed, as he said, a "Table, whose Uses are manifold, and give a more just Idea of the State and Condition of Mankind, than any thing yet extant that I know of. It exhibits the Number of People in the City of Breslaw of all Ages, from the Birth to extream Old Age, and thereby shews the Chances of Mortality at all Ages, and likewise how to make a certain Estimate of the value of Annuities for Lives, which hitherto has been only done by an imaginary Valuation: Also the Chances that there are that a Person of any Age proposed does live to any other Age given . . . ." Halley's method of constructing a life table was far superior to that of Graunt, and he has the additional merit of having computed what we call

* A paper to the Annual Meeting of the Mathematical Association, 5th January, 1937.

† *Philosophical Transactions, Giving some Account of the Present Undertakings, Studies and Labours of the Ingenious, in many Considerable Parts of the World*, vol. xvii, January, 1693, pp. 596-610.

to-day the stationary population, which indicates the number of years lived by 1000 live-born according to the mortality in the period under consideration, and which also shows the age composition of a population that with an equal annual number of births and deaths would be constantly subject to the same mortality. With all its deficiencies, his brief article contains the basic principles which we apply still to-day in measuring mortality.

The necessity of taking into account the age composition in measuring *fertility* was not realised until much later. The first who did so was apparently the Swedish astronomer Per Wargentin. At the time when he was Secretary of the Swedish Academy of Science and the moving spirit of the Swedish Statistical Commission, the Swedish statistical records began (1775) to classify the mothers bearing children by quinquennial age groups. On the basis of these records, the Secretary of the Statistical Commission, Nicander, computed specific fertility rates according to age by relating the average annual number of deliveries in 1780-1795 for each quinquennial age group from 15 to 55 years to the mean number of living females.*

Specific fertility rates for quinquennial age groups, and even for each year of age, have since been computed for an ever-increasing number of countries, but for about a century the usefulness of such rates for determining fertility was rather limited. As late as December, 1905, Newsholme and Stevenson complained that fertility rates by age "render a view *toute ensemble* almost impracticable". The problem to be solved was then to fuse the fertility rates by age into one numerical expression.

A very simple solution of this problem was submitted to the International Congress on Hygiene and Demography of 1907.† It consisted in adding up the annual fertility rates. The sum thus obtained—which has been called the *total fertility*—indicates exactly how many children, with fertility as it is, would be born to 1000 women arriving at the age of child-bearing, if none of those 1000 women died before having passed through child-bearing age.

While an adequate device for measuring mortality had been discovered in the seventeenth century, an adequate device for measuring fertility was thus found only thirty years ago, and even then it attracted very little attention. This may seem surprising to-day. One must, however, remember on what phases of vital statistics interest focussed thirty years ago. At that time, the excess of births over deaths was higher in most European countries than ever before. Mortality had decreased at an amazing rate, and since the number of *surviving* children was exceptionally large, the decrease in the number of children born was generally considered as a natural re-

* See *Kongl. Svenska Vetenkaps Academiens Nya Handlingar*, vol. xxi, 1800, p. 323.

† See Kuczynski, "Zur Statistik der Fruchtbarkeit", *Bericht über den XIV. Kongress für Hygiene und Demographie*, Berlin, 1907, vol. iii, p. 1476.

action to the decrease of child mortality. Births were of interest only in relation to deaths. In the meantime, however, conditions have utterly changed. Fertility has decreased much more than mortality, and has become, therefore, a problem in itself. The new method of measuring fertility, which ten years ago was unknown to all but a few population experts, is now used all over the world.

I began by stating that population trends are determined by fertility and mortality. Hence, in order to measure population trends, it becomes necessary to combine the best devices of measuring fertility and mortality. With this purpose in view, Richard Boeckh in 1884 calculated a fertility table.* He took the life table of the city of Berlin for 1879, multiplied the number of females for each year of age by the fertility rate for that year of age in 1879, and added the products. The sum, 2172, gave him the number of births to 1000 females on the basis of the fertility and mortality of 1879. Since according to the distribution of the sexes at birth there were 1053 male births per 1000 female births, he concluded that the real natural increase of the Berlin population in 1879 was $\frac{2172}{2053} - 1 = 6$ per cent.

Boeckh had thus virtually discovered what we call to-day the *net reproduction rate*. This rate indicates by what percentage a population constantly subject to a certain fertility and a certain mortality will ultimately increase or decrease within a generation. The net reproduction rate of Berlin for 1879 was 1·06. This means that on the basis of fertility and mortality of 1879 the average number of girls born to a newly born girl, or, what amounts to the same, the average number of future mothers born to a mother of 1879 was 1·06.

Forty years ago the net reproduction rate in England, Germany, and the Scandinavian countries was 1·4 or 1·5. This means an increase of 40 or 50 per cent. within a generation. Conditions were more or less the same in the other countries of western and northern Europe with the exception of France and Ireland, where the net reproduction rate was about 1 ; the population there merely held its own. In some eastern countries, such as Bulgaria and the Ukraine, it was nearly 2, which means almost a doubling of the population within a generation. At present the rate is below 1 in all countries of western and northern Europe, with the exception of Holland and the Irish Free State ; it is likewise below 1 in Austria, Czechoslovakia, Hungary, Finland, Estonia, Latvia, the United States, Australia, and New Zealand. It is below 0·8 in England, Norway, Sweden, Belgium, France, Switzerland, Austria, and Estonia. The only European countries in which it is above 1·2 are Portugal, the Balkan States, and Soviet Russia. Of all countries predominantly inhabited by whites, Soviet Russia is the only one where the net reproduction rate is not lower than it was in England, Germany, etc., forty years ago. In western and northern Europe as a whole the net reproduction rate has dropped from 1·3 in the 1880's

* See *Statistisches Jahrbuch der Stadt Berlin*, 1884, pp. 30-34.

to 0·8 in 1935. This enormous fall shows that fertility has decreased much more than mortality.

For most of the coloured races the available statistics are too scanty to permit a final judgment. The net reproduction rate of the negroes in the United States is apparently lower even than that of the whites. The net reproduction rate of the negroes in Central Africa is also probably below unity. The net reproduction rate of Japan seems to be about as high as it was in Germany 40 years ago.

In England the net reproduction rate for some years has been about 0·75. This means that, according to present fertility and mortality, 100 mothers give birth to only about 75 future mothers. Since a population, in the long run, can maintain itself only if 100 mothers give birth to 100 future mothers, there should have been one-third more births than actually occurred, or 800,000 births per year instead of 600,000. Yet England, with 600,000 births, had actually an excess of births over deaths amounting to over 100,000. But this excess is due only to the present age composition of the population which is not the result of present fertility and mortality, but of bygone fertility and mortality in the last 100 years. If fertility and mortality remain what they are, the population of England will still increase for some time to come. But the increase will become smaller and smaller, and deaths will exceed births long before the age composition of the population comes to correspond to present fertility and mortality. Once the latter stage is reached, that is, roughly speaking, when there will be few women of child-bearing age because women now have few children, and when there will be many old people because mortality under sixty has been so much reduced, the population with a net reproduction rate of about 0·75 will decrease by about 25 per cent. within a generation, that is by about 1 per cent. per year.

I have suggested that if fertility and mortality remain constant the population will ultimately have an age composition which corresponds to present fertility and mortality. This is by no means obvious. It is evident, to be sure, that a population with a constant mortality (for each year of age) and a constant net reproduction rate of 1 must sooner or later have an age composition corresponding to that of the life table and must from then on preserve this age composition for ever. Does it actually follow that a population with a constant mortality and with a constant net reproduction rate of 0·75 must also sooner or later have a definite age composition and preserve that for ever ?

The mathematical elements of this problem have for some time attracted the attention of both European and American mathematicians. They have come to the conclusion that a population with a constant fertility and mortality will in fact ultimately become stable. A comparatively easy approach to the computations necessary for ascertaining the age composition and the birth and death rates of the stable population is to be found in the report presented by Bortkiewicz to the 1911 congress of the International Statistical

Institute.* He starts from the stationary population according to the German life table for 1891-1900 and shows what will be the ultimate age composition, that is, the age composition of the stable population, if the population increases every year by 7, 14, 21 or 28 per 1000. He shows, furthermore, how the stable birth and death rates may be derived from that stable age composition. But since one of his assumptions, the stable rate of increase, was not and could not be based on the actual conditions presented by some specific statistical example, his findings, interesting as they were from a theoretical standpoint, did not attract the attention of demographers.

The attention of demographers was indeed only aroused when fourteen years later the American mathematician, Lotka, who for a long time had studied the theoretical problems of the stable population, published in collaboration with Dublin, the well-known article "On the True Rate of Natural Increase as Exemplified by the Population of the United States, 1920".† Unlike Bortkiewicz, who had started from a given mortality (Germany, 1891-1900) and a fictitious stable rate of increase, they started from a given mortality and a given fertility (United States, 1920) and their object was to find the stable rate of increase corresponding to that mortality and fertility. I must confine myself here to stating that they derived this stable rate of increase, by means of rather complicated formulas, from the net reproduction rate.

Unfortunately there is neither a simple nor a complicated formula which can be used for computing the development till the population has become stable. This development can only be ascertained through a special computation based on fertility and mortality rates by age. Such a computation has been carried out recently for England by Dr. Enid Charles of the Department of Social Biology of the London School of Economics. It appears from this computation that if fertility and mortality remain what they were in 1933 the population of England will drop below 30 millions before the end of this century. Conditions in England in 1933 were fairly representative of conditions in western and northern Europe. If fertility and mortality remain in western and northern Europe what they were in 1933, the population, which now is about 194 millions, would by the year 2000 be reduced to about 150 millions. If, on the other hand, the population of Soviet Russia, which is now about 175 millions, continues to grow as, according to the official figures, it has grown from 1924 to 1935, it would by the year 2000 amount to about 650 millions.

In making these statements I am, of course, far from predicting anything. My object has been merely to show what would happen

* L. von Bortkiewicz, "Die Sterbeziffer und der Frauenüberschuss in der Stationären und in der Progressiven Bevölkerung, zugleich ein Beitrag zur Frage der Berechnung der 'Verlebten Zeit'," *Bulletin de l'institut international de statistique*, 1911, vol. xix, part ii, pp. 63-138.

† *Journal of the American Statistical Association*, September 1925, vol. xx, pp. 305-339.

if fertility and mortality remain what they are to-day. This is a fundamental difference which is often overlooked. We should clearly distinguish between estimates meant to be forecasts and computations which merely show what will be the trend of population on certain definite assumptions. To predict the actual population of Europe ten years from now would involve a risk which no serious statistician should be willing to shoulder; he need only remember that quite unexpectedly the population of Europe between 1914 and 1919 decreased by 12 millions. To compute with the best available methods what would be the population 100 years from now, if fertility and mortality remain what they are, is his legitimate task if by so doing he merely wants to elucidate the present balance of births and deaths.

Discussion.

Asked whether when he said that constant mortality and constant fertility and stability were ultimately assured he meant that stability at zero was assured, **Dr. Kuczynski** said: No; I meant that, if there is a constant fertility and mortality, there will be a stable population in the sense that the age composition of the population will be stable, that the ratio of births to the population and the death rate will be stable, but of course the population will decrease or increase. If it decreases it will, theoretically, reach zero.

Asked by **Mr. Hope-Jones** for further details as to the definition of the fertility rate used as the actual criterion of increase, **Dr. Kuczynski** replied: One takes the fertility rates by years of age. You would have, let us say, at the age of 15, two births per thousand females; for the age 16, six per thousand. These would increase up to, say, 25 or 28 years of age, and the rates would then decrease up to the age of 50. You would multiply each fertility rate by the number of women surviving at this specific age according to the life-table, and you would add those products. The sum, which in the case of Berlin (1879) was 2172, included all births. As we are interested in fertility, we consider only female births, and therefore we reduce those 2172 in the proportion of female to total births. As there are 1053 male births per 1000 female births we divide the 2172 by 2053, which gives 1·06, which would be the net reproduction rate for Berlin in 1879. This net reproduction rate of course varies. The highest I ever found was for the Ukraine in 1896-97, when it was 1·96. In 1933 it was as low as 0·67 in Austria and 0·5 in Saxony, and I understand that in Berlin it was 0·3 or something like that.

Professor Neville: Is the ratio of male to female births entirely independent of the age of the mother?

Dr. Kuczynski: It seems to be. In spite of the vast literature on the subject, I do not think we have found what determines the sex ratio at birth.

Mr. Douglas White: Has any curve been determined for the relative fertility, or rather, fecundity, of women of different ages?

Dr. Kuczynski: Oh yes. This has been done for quite a time,

and of course the curve is by no means uniform. Even now the fertility of women between 15 and 25 is not much lower than in former times. For most countries, it appears that fertility of the youngest women, those under 20, is higher than in former times, but fertility of older ages has decreased enormously. In some countries, over the age of 40 it is one-quarter of what it was fifty years ago. This curve, thus, has changed very much.

The Chairman: I would like, on your behalf, to express our very hearty thanks to Dr. Kuczynski for introducing a subject which is intensely interesting and has been so much neglected in the past. I have not yet had the pleasure of reading Dr. Kuczynski's book, *Population Movements*, published by the Oxford University Press, but I hope soon to do so.

1110. THE GEOMETRY OF IDEAS.

Mme. Chantal, une grosse dame dont toutes les idées me font l'effet d'être carrées à la façon des pierres de taille, ... ; un carré, un gros carré avec quatre angles symmétriques.

Il y a d'autres personnes dont les idées me semblent toujours rondes et roulantes comme des cerceaux. Dès qu'elles ont commencé une phrase sur quelque chose, ça roule, ça va, ça sort par dix, vingt, cinquante idées rondes, des grandes et des petits, que je vois courir l'une derrière l'autre, jusqu'au bout de l'horizon.—G. de Maupassant, *Mademoiselle Perle*. [Per Mr. A. Narasinga Rao.]

1111. La guerre sur mer à cette époque se déroulait sur une seule dimension : la surface. Le développement des armes sousmarines a révélé, pendant les hostilités, que l'action navale s'exerçait dans deux dimensions. L'avenir de l'aviation l'étend à trois.—*Le Matin*, 11 novembre, 1936.

1112. "I cannot say that I owed much to any of my other teachers. One of them was the celebrated Professor W. K. Clifford, whose lectures on Applied Mathematics I attended without gaining much knowledge or illumination. At that time Professor Clifford was, of course, making his important contributions to physical science, while outside these theoretical interests he was carrying on his campaign against the Christian religion and endeavouring to set up as its substitute the service of man. Some of us thought that he did not in himself show the practical application of his doctrine, for while he was kindly he was systematically unpunctual to his lectures, and on occasion some of us after waiting till our patience was exhausted had to fetch him from the Common Room used by the professors."—J. Scott Lidgett, *My Guided Life*. [Per Mr. A. F. Mackenzie.]

1113. The condensation of metaphor involves no denial of logic: it is simply an extension of the implications of grammar, the development of a notation which, being less cumbersome, enables us to think more easily. It may be compared to the invention of a new notation, say that of Leibnitz or Hamilton, in mathematics: the new is defined in terms of the old, it is a shorthand which must be learned by patient effort, but once learnt, it makes possible the solution of problems which were too complicated to attack before. The human head can only carry a certain amount of notation at any one moment, and poetry takes up less space than prose.—*The Faber Book of Modern Verse* (1936), edited by Michael Roberts, p. 20. [Per Mr. A. F. Mackenzie.]

THE HISTORY OF MATHEMATICS: ITS RELATION TO PUPIL AND TEACHER.*

By R. M. Gabriel.

I WANT to begin with an explanation as to my position in coming here to address you. I am not an expert upon the history of mathematics. I have come in order to tell you something of my experiences in trying to teach a little of the history of mathematics to sundry university students. This had its origin in the determination of Professor Milne, even while he was teaching at Clifton, to propagate a knowledge of the history of mathematics amongst students who, as a rule, know absolutely nothing of it. He continued on that crusade in a true missionary spirit at the University of Leeds. Mr. Gilham and I have taken up the cudgels alongside him, and though we did that somewhat unwillingly, we are not willing to lay down the cudgels in a hurry.

I wish, in the first place, to comment upon the extraordinary nature of the ignorance of the history of mathematics amongst students really able and otherwise. It is possible to find Fellows of Trinity, whose powers one recognises almost as soon as one hears them speak, whose knowledge of mathematics is as wide as it could possibly be nowadays, and yet whose knowledge of the history of the subject is practically zero. The situation is repeated all the way down the mathematical ladder, and I must say that five years ago I was content that I should be a fair sample of that ignorance! I thought it rather a good thing. However, on being forced to teach a little of the history of mathematics it was not long before I came to the conclusion that there was something to be gained from this historical study. I am now firmly convinced that there are many points at which a knowledge of the history of mathematics definitely gives to the possessor of the knowledge, a point of view which can be helpful not merely in the understanding of the subject, but in the teaching of it, and I would even go so far as to say in the performance of research. It is probably not possible in the last case to put one's finger on a definite point and say that happened as a result of knowing a little history, but I am quite sure in my own case that the whole of my point of view has been changed as a result of acquiring a little knowledge of the history of mathematics.

It is typical, I suppose, that the three of us who have taken up this work of teaching the history of mathematics, should have approached it from three quite distinct points of view. I tell you that in order to indicate that it is possible to approach this matter with a very wide range of interests. It is possible to approach it with the view that history is a string of rather amusing, often salacious, stories about people who are now dead. It is also possible to approach the subject from the point of view of the antiquarian who loves burrow-

* A paper to the Annual Meeting of the Mathematical Association, 5th January, 1937.

ing in second-hand shops. The person who takes the second course will be interested in exactly when and where the minus sign first came to be used ; in the way the various notations arose in curious manners. Thirdly, I think it is possible, indeed it is my own point of view, to approach the history of mathematics, as a serious student of history approaches any history, with the object of fitting together the jig-saw puzzle which can make up some sort of a picture of the past.

With that preliminary, I pass on to discuss the difficulties of propagating any knowledge whatever of mathematical history. Since I realise that there is not likely to be much history of mathematics taught in schools until the teachers themselves know something of it, I will speak first of the difficulties of introducing the subject into a university course. The first thing that strikes one, at any rate in the University of Leeds, is that students in general, good or bad mathematicians, have extraordinarily little background of general history. History as far as the average secondary school product is concerned seems to be very largely " 1066 and All That "—more or less restricted to England. I mean by that—to take a particular example—that a student coming up fairly well armed with mathematics is likely to have little notion as to where the Greek geometers stand relative to, say, Jesus Christ and William the Conqueror. Euclid is dumped into a sort of limbo in which Nebuchadnezzar and others figure. Actual precision, there is none. I think if I gave a rhetorical question : " Did Euclid live in A.D. or B.C. ? " I should see worried faces amongst my audience.

The second difficulty in the teaching of the history of mathematics in a university is that again and again one is called upon to refer to mathematics of which the students have no knowledge. To take an example, if you want to talk about the greatness of Gauss you are largely called upon to refer to subjects of which the average student has no knowledge whatever. An average student does not know what a quadratic residue is and you have no chance of explaining.

So much for the difficulty of introducing the history of mathematics into a university course, quite apart from all the difficulties of the inertia of any curriculum when faced with the possibility of change.

Now I would like to turn for a moment to the position in schools. As the governor of sundry secondary schools I am quite well aware that in the average secondary school the pressure of School Certificate is so great that any introduction of further duties for a mathematical master is met with a look of horror. Time is so full up already with wooden methods for wooden heads that there is no room for an additional course. I was, however, pleased to notice that when this matter was first mentioned in a small group of council members, that a mathematical master from a very distinguished school took me up somewhat sharply, when I made a remark of that kind. He said that if there was no time for a little mathematical history in a school course, there was something wrong somewhere. I took that as an

indication that in schools, where a bunch of good talent is likely to be congregated amongst pupils, there probably is time for the introduction of some history of mathematics without there being a corresponding sacrifice of mathematical teaching proper. I agree, of course, that something is wrong, but it is not likely to be put right in the majority of secondary schools.

The other difficulty is, I think, one which is not likely to be overcome in any case ; that is, that as far as secondary school work is concerned the scope of mathematical history is necessarily restricted and in all probability is so small that only in the VIth Form can anything in the nature of a connected account of any period be introduced with advantage.

Having sketched out the difficulties of introducing this subject at all, I might indicate at this point that I consider that throughout the course in the school, even under the most terrible pressure of examinations, it is possible to introduce slight references here and there which will make the names connected with mathematics something more than mere names. Pythagoras, so far as schools are concerned, has a theorem, but he has very little else except possibly some connection with Shakespeare ; I think he crops up in *Twelfth Night* in connection with the transmigration of souls. To take that as a specific instance, I definitely consider that in the teaching of elementary geometry in schools it is possible to introduce some odd remarks as to how much Pythagoras knew, and when he lived; what his interests other than the geometrical ones were. The account of his theory of musical tones, for instance, is one which I am sure can be explained in a rough and ready way, without going into detail, to a class, and it makes Pythagoras something more than the mere name which occurs to me as I visualise one of my masters walking into a room, banging on the desk, and saying " Write out Pythagoras ". That situation can be altered and, as I shall show later on, it is worth altering. The effects upon the general liveliness of the subject are quite noticeable as a result of the discussion of a little history of mathematics.

I propose to enumerate some special points which have struck me with pleasure in the course of some three or four years' work that I have put into this subject. When I began to lecture on the history of mathematics I confess that I looked upon it as the recapitulation of a number of mathematical facts, and as being totally cut off and totally distinct from ordinary history. I very soon had to realise that mathematical history is in no sense cut off from the ordinary type of history which is taught in schools. It is as much a reflection of life and the time as is any social history. It does not tell you, of course, the details of the life of a mass of people, but if one has a background of a particular period in mind, the mathematical history fits in extraordinarily well. For instance, I am not suggesting that the works of Newton will give you an account of the life of the seventeenth century, but I maintain that if you have any notion whatever of what the seventeenth century stands for, you will say

with regard to Newton that he clearly and undoubtedly belonged to that extraordinary period of thinking between 1650 and, say, 1720. It is possible to recognise a man's period in his work, and I find it a most interesting thing to do. Another example, of course, is the work of Cardan and Tartaglia. They belonged to the Italian renaissance. I am not suggesting that a knowledge of the history of mathematics of those people will tell you all about the time in which they lived, but they do fit into the general picture of their time in a way which is perfectly delightful to behold. To come to the nineteenth century, people of the Kelvin type with their peculiar materialistic outlook, form a picture which makes up a whole ; it is not a series of disconnected fragments.

That was a very surprising discovery to myself. How far it is possible to communicate to rather immature people of 19 to 21 the way in which the mathematical work fits into the picture of a period as a whole is perhaps another matter. I am quite well aware that I have not been very successful, but to me it was a very pleasant thing to stumble upon that.

Equally, the characters of the men engaged reflect themselves. One forms the impression, quite rapidly, that Newton must have been a very very decent sort, though shy and somewhat peculiar. Equally, one forms the impression that Cardan must have been a perfect pest, and so on. The attitude of Descartes, for instance, and of Laplace is one which would be very trying in a common room ; and so the view that the history of mathematics is merely a chronological collection of the occurrence of theorems is completely fallacious. That view left me before the end of some six months' work.

It is of importance that schoolboys, schoolgirls and students should get a more satisfactory view of the nature of men whose work they study than they have at present. I am aware, for instance, that schoolboys, as a rule, have little notion of the strangeness of the great mathematicians. They would probably get a distinct shock if they discovered it, but I think a knowledge of the nature of the mathematicians would add to the subject an interest which is at the moment lacking. Examples which occur to me straightaway are those of Kepler and Sylvester. That Kepler after a first unsuccessful marriage solemnly wrote out the advantages of the eleven possible candidates for his second marriage, and of these on mathematical principles chose the very worst is a point of interest. Sylvester's poetry should warn students that they must not expect of mathematics a basis for an outlook on life. To do them justice, schoolboys and students run very little risk of doing that! But I do find it extremely awkward to deal with a set of students who come up taking mathematics for the reason that they could not think of anything else to do or because the headmaster or headmistress has told them to do it. I think that they have practically no chance of forming a correct notion of what mathematics is about and why it is studied, in the ordinary school course, and a knowledge

of the history of mathematics might make the matter appear a little more in its proper light.

I want to enlarge that point a little. It is usual in teaching mathematics to present the subject in its finished form, in which the fourth or fifth theorem follows directly from those that have gone before and, as a rule, enormous stress is laid upon logical development. That is probably a relic of the last century point of view of mathematics as what is called " mental discipline ". As such it is largely a failure. I am under the impression that if we depart from that attitude, even in schools, we gain something. And there is one thing above all others that the history of mathematics reveals, and that is that the development of mathematics is usually upon anything but the logical finished and polished forms in which the stuff is learned. The ordinary schoolboy's view of Euclid is that Euclid sat down and, first of all, wrote Book I, then Book II and then Book III in correct sequence ; that he never thought of the Theorems until he had proved them, and certainly never used them. Nothing could be further from the origins of the work, and I noticed in what little school teaching experience I had years ago, that this notion of not using a result which seems obvious, until you have actually proved it, is one totally foreign to the schoolboy mind ! The history of mathematics is, upon the whole, on the side of the normal schoolboy point of view.

Now I suppose that as a schoolboy and a student I must have been a rather inquisitive child and young man, asking many questions that perhaps were not usual. I made a perfect nuisance of myself at school asking why it was that in certain propositions concerned with similar triangles only rational ratios were dealt with first of all, and then amidst a rather murky mess we passed to the general ratio. I definitely asked the master in charge why this was, why he did not go straightaway to the general ratio. The only answer I got was a threat of punishment. The reason why I did not get an answer was, I think, simply that the master in question did not know the answer. Had I been dealing with my senior mathematics master I should have been more fortunate. But I suggest that the reason why that peculiar form enters into the Euclidean discussion of ratio lies essentially in the history of Greek mathematics. It goes back, in the long run, if I am not mistaken—and if there are any experts upon the history of early mathematics here they are probably in a better position to speak about it than I—it goes back to the nature of Pythagorean number and the subsequent discussions of the incommensurables which go on through Zeno and continue up to our present time. A little knowledge of the history of that in the master is likely to prevent the brick wall against which I ran in my questions.

The same thing applies to the position with regard to the definition of point and line. We were made to write down at the word of command that a point was that with position but not magnitude; that a line was a thing with extent but without breadth. And then we cheerfully went on to material proofs by superposition. Now, I

was very puzzled as a boy why certain proofs by superposition were allowed and certain others were not. Again I was fool enough to ask the same master why ! This time I *did* get into trouble. I am quite sure that particular master, whose attitude really was that Greek geometry was a method of making children behave themselves—that is to say, it was an excellent chance for tripping them up when they did anything they had not been told to do—had no notion whatever of the Greek point of view with regard to the ideal corresponding to the chalk line, the line in the sand and so on. He had no notion of it whatever.

A little knowledge of the history of Greek mathematics, fringing admittedly on Greek philosophy and Plato, lies, I think, at the bottom of why the definitions are cast in that particular form, and I am quite sure that a knowledge of Plato and, to a certain extent, Socrates gives an indication of why the whole of Euclid is cast in its particular form, why a formal proof with its definite stages is put with such remarkable emphasis. After all, it is a very remarkable thing that there is no other civilisation in which you will find that particular type of thinking. This is a point that is worth making. You can look through the Old Testament. You can look through the early writings of almost any civilisation and you will not, I think, find the mental abstraction which is put down and for ever perpetuated in the Euclidean ideas.

I have given you some examples in the course of what I have said of the sort of thing from which I think definite advantage is to be obtained for the teacher in his outlook, and of course that inevitably affects his power of teaching afterwards, assuming from the start he knows how to keep order and get work done.

I want to name now a few particular topics which, it seems to me, are well worth studying by anyone concerned with the teaching of mathematics. One of them is undoubtedly the early history of logarithms. It is as distinct from what you might call the logical development theory of logarithms as anything can be. I think the ordinary school course on the logarithm presents itself as a piece of abstruse thinking by people with no particular practical point of view in mind. We start with indices and play with them and then there comes the doubtful extension to the actual logarithm. What actually happened was that Briggs, an English mathematician, and Napier, a Scottish mathematician, wanted to do heavy calculation. The ever-extending work of the sixteenth century demanded that calculation should be more rapid in dealing with heavy figures. So far from landing on the final simple form of a logarithm the two pioneers made themselves many unnecessary difficulties at the beginning. I commend that to anyone interested in the teaching of mathematics, and I suggest that in schools where you have VIth form boys with some small ability that a paper upon this early history is quite within the powers of a VIth former. The ideas involved are not heavy. There is quite a good account of it in Cajori's book (*A History of Mathematics*), and also, I think,

what is in the *Encyclopaedia Britannica* is sufficiently intelligible for a good VIth former. If you want a story of even more practical interest there is the story of the consequences of the discovery of logarithms. Briggs and Napier made it possible for calculations involving seven and twelve figures to be done with facility, and about the same time Tycho Brahé had made his observations. Coupling these people's work together with extraordinary patience and faith, Kepler showed that the orbits of the planets are ellipses. Then the final phase comes in the story of Newton finding his mathematical descriptions of the situation, upon which all his ideas of dynamics were founded. These things are well within the grasp of any average VIth former. I am not restricting that to brilliant scholarship men. It is the sort of thing which can be handled by less able pupils.

For persons more interested in physics and who wish to get away from actual manipulation I recommend the ether theories of the last century, an amazingly interesting study ; and, more advanced still, there is the history of the extension of Newton's particle dynamics up to, say, Hamilton.

Another point which is of extreme interest and is within the powers of the VIth former, though probably it would not interest the non-mathematical people so much, is the history of the development of Cartesian coordinates. The common, rather journalistic, account which I think I had thrust upon me as a child, that " a French king said he was bored and wanted something new, would Descartes please think of something for him and Descartes thought of squared paper ", is about as far from the truth as it can be. A little study of the true history is well worth while. It reveals that Descartes saw the true inwardness of his own work and laid his stress upon the fact that the equation contains *all* the properties of the curve. Relatively little stress is laid upon the axes themselves.

I will sum up the products in my mind of my last three years' work upon this subject. I believe that the impression of mathematics produced by the ordinary formal course in school and university is a thoroughly false impression. The presentation of mathematics in its final and polished form gives an unfortunate impression which is in no way necessary and definitely does not correspond to the point of view of the actual pioneers in mathematics. The reason of the absence of reference to the history of mathematics in schools is, in the main, the ignorance of the subject which follows inevitably from its exclusion from the universities. In spite of their obvious deficiencies, the histories of mathematics now on the market—Rouse Ball, Cajori and, later, Smith—provide a teacher with a sufficient knowledge of mathematical history, for the teacher in turn, to give some sort of setting to our subject to his or her pupils. In addition to the absence of knowledge there is, undoubtedly, the difficulty that you do not get much chance of a coherent formal course in the history of mathematics before the VIth form stage is reached. I propose to sit down at that point.

DISCUSSION.

The Chairman : I would like, at once, to thank Mr. Gabriel for his extremely interesting and humorous paper. It is the most stimulating paper we have had for a long time. We have with us to-day a distinguished mathematical historian in Sir Thomas Heath and I hope he will be moved to speak.

Mr. Gabriel has rather emphasised the use of a continuous course of mathematical history and he has referred to the VIth form, but the most valuable place for the mathematical historian is below that form. I see, of course, that Mr. Gabriel wants to get the teacher to get some history, and I am going a step further. When the teacher has got that history I want him to introduce pieces of history lower down.

I recall an incident of which I venture to tell you. Some twenty-three or twenty-four years ago there was a knock on my door a little after 10 on a Monday morning. I found outside the headmaster who wanted to introduce me to an American who wished to watch me teaching. I was discussing variation and telling the boys about one quantity varying as the square of another. I remarked, " Supposing I have a stone and let it go ; it will fall a certain distance in a second. It will fall another distance in two seconds ". I referred to getting the variation of one quantity as the square of another, and a little boy said : " Please, sir, do stones all fall at the same rate ? " I said : " Yes, Galileo settled that when he dropped a 1 lb. weight and a 100 lb. weight from the top of the Leaning Tower at Pisa. He announced, first of all, that they would fall together, and he assembled the University of Pisa to watch this, as Oliver Lodge describes in his book, *Pioneers of Science*. The weights fell to the ground together, and that was the death-knell of some of the old beliefs about the behaviour of bodies. By the way, look that up in Lodge's *Pioneers of Science*." I went on with the lesson and at the end of it I had some chat with the American. He said, " I guess no teacher in my country would have dared to wander as far from his subject as you did this morning." He then went on to pay me a compliment. " But you were precious quick over it, and the time that you took on it, I guess, was more than made up by their added interest." Then he paid me another compliment, one of the greatest I have ever had paid me in my life. On the next Monday morning at exactly the same hour there was a knock on my door and the American was there again. He said, " I guessed you would be teaching that same class again. May I come in and watch ? " I think I am right in saying that Mr. Gabriel is wanting to get that sort of thing into the teaching.

I referred to Lodge's *Pioneers of Science*. That book is on the shelf in my room and every boy in the classroom knows it. Having referred to it in a lesson, the next time I go into my room I am disappointed if I do not find a boy looking this up and asking if he could borrow the book. I am quite certain that that sort of thing has an enormous influence on the interest people take.

I always tell boys to remember that Newton was born in the year in which Galileo died, and the moment I have said that I tell them that it is not strictly true, but it is near enough. I say that at once because I daresay I shall be corrected if I do not mention it. I tell them the story of Milton going as a young man on the grand tour to visit Galileo and sympathising with him in his blindness and finding Galileo's daughter acting as his secretary or, as the books say, his amanuensis. When talking to the boys I have not up to that point mentioned Milton's name ; I have told them of a " brilliant young Englishman " who did that, and then I tell them that that brilliant young Englishman was John Milton. That makes an impression. Then I ask : " Where do you put Milton in history ? " They put Milton somewhere at the Commonwealth. " All right ", I say, " that's the middle of the 1600's." " Very well, then, what did Galileo do and what did Newton do ? What about Descartes ? What have we got that is named after Descartes ? Our Cartesian coordinates. Did Descartes live before Newton or after ? Could Newton have invented the calculus if there had been no Cartesian coordinates ? " Little points like that brought in, as the American said, without wasting much time over them, do add enormously to the interest of children in the subject and act as a stimulus and, I believe, make mathematics a much more human subject. We get much more out of our pupils if we do that sort of thing. It is a subject about which I am keen. I do believe in bringing in, but not wasting time over, mathematical history in early teaching. I give boys sums to do in Roman figures, addition, multiplication and so on. I am extremely interested in Mr. Gabriel's paper, and I hope that Sir Thomas Heath may favour us with a few remarks, also some other members of the audience.

Sir Thomas Heath : It is most kind of Mr. Siddons to refer to me in this connection, though I was not prepared for it, and I find it a little embarrassing because it is not easy to say anything useful on such a subject on the spur of the moment. I have, of course, spent a great deal of my time on the history of mathematics and especially Greek mathematics, and I may say that I have never regretted it, for I have always found it a real " stand-by " and a source of abiding interest and pleasure. The study is fascinating when you first take it up, and its lure grows and grows. I am still working at Greek mathematics after more than fifty years and, in particular, on the mathematical passages in Aristotle and the Aristotelian corpus. It is this hobby which I think has kept me alive. I depended upon it as a relaxation during the long years when I worked at the Treasury day in and day out. The strain in the last years, especially the war years, was great and but for my hobby I feel that I should hardly have kept going.

I have listened with great interest to the lecturer and to the remarks from the Chairman, with which I much sympathise. I would conclude by thanking him for alluding to my presence, and by congratulating Mr. Gabriel on his interesting paper.

Mr. G. L. Parsons (Merchant Taylors'): Perhaps I might be permitted to add one or two topics, of rather less advanced character than those mentioned, in which the history of mathematics can play a part. One of these is " completing the square ". It is significant that the method of solving quadratic equations, which was the very first one ever used, has recently reappeared in modern textbooks, but I have never seen it put there with the note that it was actually the way in which Muhammed Ben Musa solved his first quadratic equation. Another point is that when trigonometrical ratios are first introduced many will want to know why these things are known by such curious names (sine, cosine, etc.), and you can draw them that picture of the arc of the circle, the bisector, the two radii, the chord of the arc, and slip in the tangent afterwards. I believe that in the older trigonometries the arrow, *sagitta*, actually was there. The sine is, of course, the bow-string, and that rubs the point in. While you are doing that or at some other time you get a splendid opportunity of talking about the flow of knowledge from West to East and back again, how Alexandria was burned down, of the way in which on the break-up of the Roman Empire the centre moved to Byzantium and how the Caliph Almamun in about 800 summoned some Greek sages to Baghdad where they brought with them works of Euclid, Apollonius and some of the other Greek philosophers, which were translated into Arabic and are preserved to this day inasmuch as the only record we have of some of the books of Apollonius is in the Arabic translation then made. Another place in which mathematical history must come in (with this same topic of the travel of knowledge from Alexandria to Byzantium and thence to Baghdad and so on) is in the derivation of the word Algebra, which many children will no doubt be interested to hear.

In these days many children go abroad for their holidays, and not a few pass through that old-fashioned town of Bruges in Belgium. You can tell them that as they go down from the station to the square they will see on their right hand, in a little square, the statue of Simon Stevin, or Stevinus as we call him in statics, with his experiment of the endless chain and the right-angled triangle on the front of the statue. That is not a bad way to start talking about the law of the inclined plane.

Then, again, there is the interest of the Pythagorean school in the mystic properties of numbers. They were very interested in that type of thing and invented names for all manner of different sorts of numbers which survive in the Algebra books. It is not hard to make that into a beginning of a study of progressions and series.

Before sitting down I would like to mention one more advanced topic, which is the correlation of those various definitions of the conic which appear in textbooks, one geometrical, the other as a section of the cone, and the relations with the general equation of the second degree.

Might I add to Mr. Gabriel's list of books one by Dr. Sanford, *A Short History of Mathematics* (Harrap, 1930), and a smaller one, *The*

History of Mathematics in Europe (Oxford, 1925), by J. W. N. Sullivan.

Mr. M. Black (University of London, Institute of Education): I should like to add a small bibliographical note. There are two books I have in mind, one of which has appeared quite recently and may be unknown to many here, *An Introduction to the Study of the History of Mathematics*, by George Sarton,* who is one of the best historians of science. The book has the best bibliography on the subject that I have yet seen. It is a slim book and you may feel that 6s. 6d. is a large sum to pay for it, but I can assure you from a personal inspection that it will be found very helpful. The other book which I discovered and have read recently is by P. Boutroux and is entitled *L'idéal scientifique des mathématiciens*. It contains a most lucid account of Greek mathematics and two further sections, dealing with the seventeenth century and with quite recent mathematics.

Professor E. H. Neville (University of Reading): The sub-title of Dr. Sarton's book is *The Substance of the Inaugural Lecture of a Course on the History of Mathematics in Harvard University*, but the course seems to have been rather wider than that description would imply. Do not be put off by finding that the introductory lecture contains only the sort of generalities which anybody might put forward. The bibliography which follows forms two-thirds of the book and is quite uniquely valuable. It is annotated from Dr. Sarton's own reading. The books he deals with are only those he has handled personally so that we have the advantage of a direct first-hand impression of the books and not merely repetition of what it has become customary to say about them. On the whole, that independence outweighs the fact that, naturally enough, there are some judgments with which other readers would not agree. It is an expensive book, but to anyone who wants bibliographical guidance it is to be recommended warmly. Turnbull's book, *The Great Mathematicians* (Methuen, 3s. 6d.) should be mentioned here to-day; of unpretentious introductions it is possibly the most interesting to those who like history tacked round biography rather than dealt with in the Hogben manner.

Mr. Gabriel: You must have noticed from the speeches which have followed my own that my limitations are essentially those of a person handling university students. I am sure that from the point of view of school work the books which have been named will be admirable. But I am sufficient of a mathematician to be unable to keep quite quiet on the subject of what I think about these books. It seems in the main as if the American writers upon the history of mathematics had an insufficient understanding of mathematics. I am sorry to say that the pioneer Rouse Ball often commits to paper judgments which I am really rather surprised to find. In fact, in regard to these popular textbooks it is necessary to adopt the critical attitude.

* Published by Harvard University Press. (Reviewed in the *Gazette*, February 1937, pp. 71-72.

TRIAL AND ERROR AND APPROXIMATION IN ARITHMETIC.*

By A. C. Aitken.

Some years ago there was a vogue for a series of books dealing imaginatively with the future and bearing such titles as *Daedalus, or the Future of Science*, *Icarus, or the Future of Flying*, and the like. I do not recall whether arithmetic was among the topics considered by the ingenious authors, but a further title, *Archimedes, or the Future of Calculation*, might certainly have been suggested by the developments of modern arithmometers.

One commonly hears the remark that the increasing reliance on calculating machines will render unnecessary, sooner or later, ordinary proficiency in arithmetic. In my own opinion this is not likely to happen. Long and tedious computations with pen and paper will of course be done away with ; these can safely be entrusted to the machine, which revels in divisions and in cumulative multiplications. The pen-and-paper arithmetician will become the operator of a machine ; but his directive capacity will be very greatly increased if he has at his command a stock of approximative methods, both for general procedure and for checking.

If time permitted and a machine were available, it would be interesting to show how wide a gap may lie between the derivation of a formula for solving some type of problem and the actual practice on the machine. Professional computers have reason to know that the beauty of many theoretical solutions is skin-deep ; they know also that many formulae which look uncouth on the printed page become simple and idiomatic when translated into the quite different language of the setting levers.

I propose this afternoon to demonstrate, to a certain extent by mental arithmetic, some examples of simple methods of approximation, by no means novel, applied to the commonest standard problems, such as division, the evaluation of reciprocals and more generally of decimal fractions, squares, square and cube roots and other fractional powers, and so on. A desire has also been expressed that I should conclude by saying a little on the subject of arithmetical memory and the arithmetical faculty.

For expert work a computer will require a certain numerical equipment. It would not be unreasonable to demand that he should carry in mind the multiplication table up to 25 times 25 ; the squares of numbers up to 50 ; the reciprocals, to five or six significant digits, of integers up to 20 and of halves of some odd integers ; possibly also the square roots, to the same degree of accuracy, of 2, 3, 5, 6, 7, 10, 11 and 13. To know the logarithms of these numbers is of no great value ; for many kinds of calculation the machine has superseded the table of logarithms. A useful

* A paper to the Annual Meeting of the Mathematical Association, 5th January, 1937.

addition is to know the factors of some numbers ending in 99 or 01, for example $299=13\times23$, $301=7\times43$, $399=7\times57=19\times21$, $799=17\times47$, $801=9\times89$, $899=29\times31$, $901=17\times53$, $999=27\times37$, $1001=7\times143=11\times91=13\times77$.

The essence of a good approximative method lies in the way in which each approximation generates the next. We can often arrange the steps of computation in such a way that an unknown number x is expressed in terms of a small convenient submultiple of itself, such as $x/10$, $x/100$, or the like. In such a case each digit of x, as it is obtained, is inserted into the $x/10$ or $x/100$, ..., and the lag of one or more digits allows a uniform arithmetical process to be developed.

As a purely illustrative example, let us find the decimal for $\frac{26}{43}$, by multiplying numerator and denominator by 7.

We have $$x=\frac{182}{301}=\frac{182-x}{300}.$$

A first approximation is 0·6; hence we try $\frac{1814}{3000}=0{\cdot}6047$,

which gives the better $\frac{18139\,53}{3000000}=0{\cdot}604651$,

" " " $\frac{1813\,95349}{3000000000}=0{\cdot}60465116$, and so on.

It will be found instructive to evaluate $\frac{17}{23}$ in this way, in the form $\dfrac{221}{299}$, or $\dfrac{221+x}{300}$.

This leads of course to a simple way of dividing by a number of the form $k.10^n\pm1$, where k does not exceed 12, by *short* division. In the one case, where the divisor ends in 9, we *add* the quotient, digit by digit as it is obtained, to the dividend, displacing it the required number of places to the right; in the other case, where the divisor ends in 1, we *subtract* the quotient from the dividend, that is, we write "nines' complements". In justification all that needs saying is that we are replacing

$$m=nq+r \text{ by } m\pm q=(n\pm1)q+r,$$

while using the fact that each digit of q obtained by dividing $m\pm q$ by $n\pm1$ is found one or more stages earlier than it is required in the dividend.

Examples. $\frac{37}{59}$, $\frac{37}{61}$. The displacement is of one digit.

$$\frac{6/37{\cdot}6271186\ldots}{{\cdot}6271186\ldots} \qquad \frac{6/36{\cdot}3934426\ldots}{{\cdot}6065573\ldots}.$$

In the second example here we are really writing 37 as 36·9999..., in readiness for subtraction. Let me digress here to remark that though our Western system of numeration is a good one, it might still be improved. One defect is that we have no *single-digit* symbol for 10, to use as an alternative for the double-digit 10. No computer would ever subtract 634,197 from 2,500,000 by working from right to left, with "borrowings"; he would work from left to right, using nines' complements. On occasions like these we *think* in the

form "nine hundred and ninety ten"; we should also both *say* and *write* it, in suitable symbols. The Japanese abacus, the *soroban*, has these alternatives not only for 10 but for 5, greatly to its advantage.

[Another disadvantage is our scale of notation. The radix 10 was originally adopted for anatomical reasons. It is true it is not such a *damnosa haereditas* as 7 or 11 would have been, but either 8 or 12 would have been preferable.]

Example. The recurring decimal for $\frac{1}{67}$. The multiplier 597 is found digit by digit from the right.

67	4) 0597
597	0149
469	2537
603	3134
335	3283
$39999 = 40000 - 1$	5820
	8955
	2238
	8059
	7

The displacement being of 4 digits (because of the four zeros in the divisor 40000) it is convenient to write the quotient in groups of four digits directly under each other. Division proceeds until the digits 0597 recur, without remainder. The period is then complete.

It will be instructive to try $\frac{1}{17}$, $\frac{1}{47}$, $\frac{1}{61}$ and $\frac{1}{97}$ in this way.

For squares, square roots, cube roots, and so on, no computer would ever look beyond Barlow's Tables, in Dr. L. J. Comrie's new edition. At the same time it is very good practice, especially if one has access to a machine, to dispense with tables and try various approximative methods.

Squaring offers no difficulty. One can square numbers mentally by using the identity

$$x^2 = (x+y)(x-y) + y^2$$

with y so chosen that $x+y$ or $x-y$ is a simple multiplier.

Examples.

$$387^2 = 4\,..\,\times 374 + 13^2$$
$$= 1496\,..\,+169$$
$$= 149769.$$

$$2387^2 = 25\,..\,\times 2274 + 113^2$$
$$= 5685\,...\,+126\,..\,+169$$
$$= 5697769.$$

We do not burden our mind with final zeros in a product; blanks are registered instead, to be filled by some later addend. I have known boys at school able to square 3-digit numbers in this way, and with practice one can do 4, 5 or more.

[At this stage the numbers 319, 426, 1814, 2419, 3227 were offered from the audience, and squared mentally.]

Square root, being an inverse process, is more interesting. Hor-

ner's method can be used on the machine, and is very good if we begin well with, let us say, three correct digits, for then we can proceed in groups of three digits at a time, doubling the latest group in the divisor between each division. Of other methods an excellent one is Newton's, namely : *take an approximation to the square root, divide it into the number, and take the average of quotient and divisor as a better approximation.* If h is the (small) relative error of the first approximation, the second is in excess of the true square root by a relative error nearly equal to $\frac{1}{2}h^2$. (As a mnemonic, note that this is the integral of h.) Thus the accuracy can be estimated and we can, if desired, either apply a correction or go on to a second approximation. The whole point is to obtain a good first approximation.

Example. The square root of 80. Here 9 and $8\frac{8}{9}$ are a good first Newtonian pair. Their average is 8·9444444 .... Since they are in ratio 81 : 80, the relative error h is close to $\frac{1}{161}$, and so $\frac{1}{2}h^2$ is about 1 : 52000. The correction is therefore about −0·00017, and an improved approximation is 8·94427, which is actually correct to six significant digits.

For obtaining first approximations one may use various devices. One which I occasionally use for amusement depends on the fact that

$$(1+4h)^{\frac{1}{2}}=\frac{1+3h}{1+h}+\frac{2h^3}{(1+h)(1+3h)} \text{ approximately,}$$

provided h is fairly small. The first term on the right gives a result which is itself much superior to the first two terms $1+2h$ of the binomial expansion, and the addition of the second term greatly improves the result.

Example. The square root of 1·44. The first term gives

$$\tfrac{133}{111}=1{\cdot}198198\ldots,$$

the relative error being 1 : 666. With the second term added, we have

$$\frac{133}{111}+\frac{2\times 1331}{111\times 13300}=\left(12\frac{1}{738150}\right)\Big/10,$$

which is in excess by a relative error of 1 : 8857800. In this example the original h is by no means small.

The approximation used here is a particular case of a more general one for the pth root of $1+2ph$, namely

$$(1+2ph)^{\frac{1}{p}}=\frac{1+(p+1)h}{1+(p-1)h}+\frac{2}{3}\frac{(p^2-1)h^3}{(1+p-1h)(1+p+1h)} \text{ approximately.}$$

It will be found instructive to try the cube roots of 1·060, 1·120 and 1·150 with this.

The following method of improving a first approximation is of great power, and very well suited to the machine. Let m be a first approximation to $\sqrt{n}$, so that m and n/m are a first Newtonian pair. Let us *quarter* the interval between them (or between any convenient

multiples of them) and take the "quartiles", $m+\frac{1}{4}\left(\frac{n}{m}-m\right)$ and $m+\frac{3}{4}\left(\frac{n}{m}-m\right)$ or equally well $\frac{n}{m}-\frac{1}{4}\left(\frac{n}{m}-m\right)$, as *weights* in a weighted average of n/m and m. [Thus for the weight of each we take the remoter quartile, in a kind of chiasmus.] This weighted average is a greatly improved approximation to $\sqrt{n}$.

Example 1. To find $\sqrt{2}$, taking $1{\cdot}4$ (or $\frac{7}{5}$) and $1{\cdot}4\frac{2}{7}$ (or $\frac{10}{7}$) as a first pair. These are in ratio 49 : 50 or 196 : 200, so that the weights are 199 and 197. Since the common digits $1{\cdot}4$ of the first pair must be reproduced by any weighted average whatever the weights, 199 need not enter into the calculation. Hence we have

$$1{\cdot}4\frac{2\times 197}{7\times 396}=1{\cdot}414213564\ldots,$$

whereas

$$\sqrt{2}=1{\cdot}414213562\ldots.$$

Example 2. Taking $\frac{99}{70}\left(=1{\cdot}414\frac{14}{99}+\frac{100}{693}\right)$ and $\frac{140}{99}\left(=1{\cdot}414\frac{14}{99}\right)$ as a first pair, their ratio being 9801 : 9800 or 39204 : 39200, we have at once

$$1{\cdot}414\frac{14}{99}+\frac{3920100}{693\times 78404}=1{\cdot}414213562373095050,$$

whereas

$$\sqrt{2}=1{\cdot}414213562373095049 \text{ to 19 digits.}$$

The process used is an adaptation for machine (or even for pen and paper) of two successive steps of Newton's process. The error it commits is very nearly $\frac{1}{8}h^4$; if it is applied twice (once is usually enough) the error is $h^{16}/32768$.

You will by now have seen that all I have been doing is to dress up old material in new trimmings. In leaving these topics and turning to matters which are perhaps of greater interest, I should like once again to emphasise the great value of processes rather than of formulae, and of methods of iteration in which approximations, even though rough at first, serve towards their own improvement.

If I now conclude with a word or two on the nature of the arithmetical faculty, let it be understood at once that any personal reference is made from an entirely objective viewpoint. It is true that years ago, during adolescence, I devoted myself with some pains to the improvement of this faculty; but I regard myself as hopelessly superseded and outclassed by the machine. My inclination towards mental arithmetic came in the first place (you may be interested to hear) not from arithmetic, which I found tedious and repetitive, but from algebra; in fact it arose on the day when the teacher pointed out how to square numbers by using the identity which I have mentioned earlier.

What factors enter into mental calculation and memory? The psychologist Binet, in 1892, had unusually good opportunities for examining this problem. He had under experiment at the same time two calculators, one, the celebrated Inaudi, of almost purely auditory

type, the other, Diamandi, a visualiser; and his conclusions (in which I do not wholly believe) are well worth reading. My own experience, both in calculation and in remembering, is that visual and auditory elements are certainly present, but as *effects*, not as causes; they are set in motion by a dynamic factor, rhythmic in nature, embodying the will to do whatever is proposed. For example, if I summon up the recurring decimal for 1/97 (as I never do except on occasions such as the present), I observe that it always falls into phrases of fixed though by no means of equal length, as follows:

010309, 2783505154, 6391, 75257731, 9587628865979381, 4432, 9896907216, 49484536082474, 22680412371134020618556 7.

Each phrase begins with an up-beat, and there is an accent on every alternate digit; a visual image moves along also, like a spot-light not quite synchronised, and weaker in the middle of a long phrase. A comparison with the memory for music is apt, and will occur to you; there one does not, except as a last resort, visualise the stave. A comparison with the memory for poetry might also be made; but there the meaning (I am referring to poetry that has a meaning) partly releases the rhythm from its dynamic function and allows it to perform its proper function of integrating words of deep emotional or allusive significance into the effect of incantation.

To prove the weakness of visualisation unaided by rhythm, a sufficient experiment for me is to try to remember the decimal for 1/97 from the end back to the beginning, in reversed order; and as you see [here a demonstration was given] it runs haltingly. To prove the strength of rhythm we may take Shanks's value for π, arrange the decimal part in groups of 50 digits, each group of 50 in sub-groups of five, and subject each group of five to the metre $\smile\smile\smile\smile-'$. As you see [here π was written down], it runs easily enough. [The chairman was satisfied with 200 places.] A. C. A.

1114. These effects are not felt by every reader: to some, the devices are merely evidence of technical incompetence. It is, however, demonstrable that some people respond to them without having them pointed out: the only possible conclusion is that these people are more sensitive to language than others. The only objection to such devices is that it would never be possible to teach everyone to respond to them, therefore they tend to cut off one section of the community from another. But the same objection could be brought against the theory of tensors, and it is as necessary that some members of the community should explore the possibilities of language and use it to control and clarify emotional, spiritual and sensuous experience, as it is that others should use their mathematical notation to codify and organize our scientific knowledge.—*The Faber Book of Modern Verse* (1936), edited by Michael Roberts, p. 31. [Per Mr. A. F. Mackenzie.]

1115. More often than prose or mathematics, poetry is received in a hostile spirit, as if its publication were an affront to the reader.—*The Faber Book of Modern Verse* (1936), edited by Michael Roberts, p. 1. [Per Mr. A. F. Mackenzie.]

ENGLISH AND FRENCH TEACHING METHODS COMPARED.

A study based on *Advanced Trigonometry* by C. V. Durell and A. Robson.*

By Jacques Devisme.

The aim of this article is to convey the impressions received by a French Professor of Mathematics on reading the book referred to in the title, and thus to throw some light on similarities and differences between methods of teaching in the two countries.

In order to facilitate the making of comparisons, I have prepared this article with the following French books at my elbow :

E. Blutel : *Leçons de Mathématiques Spéciales* (2de edition. Hachette ; 1923).
H. Commissaire et G. Cagnac : *Cours de Mathématiques Spéciales.* (Masson ; 1936.)
R. Garnier : *Cours de Mathématiques Générales.* (Gauthier-Villars ; 1932.)
G. Illiovici et A. Sainte-Lagüe : *Cours d'Algèbre et d'Analyse.* (Librairie de l'Enseignement Technique ; 1933.)
Th. Leconte et R. Deltheil : *Eléments de Calcul différentiel et de Calcul intégral.* (Armand Colin ; 1926.)
B. Niewenglowski : *Cours d'Algèbre.* (10e edition. Armand Colin ; 1924.)
G. Papelier : *Précis d'Algèbre, d'Analyse et de Trigonométrie.* (9e edition. Vuibert ; 1923.)
E. Vessiot et P. Montel : *Cours de Mathématiques Générales.* (Librairie de l'Enseignement Technique ; 1921.)
M. Weber : *Algèbre et Trigonométrie.* (Dunod ; 1925.)

This list has not been compiled solely on merit, for there are many other excellent books which do not figure in it ; I have taken those mentioned for purposes of comparison simply because they were at hand while I was preparing this article.

The first impression concerns the actual presentation of the text : very short sections, two or three pages on the average, each develop only one idea ; these are all followed by very numerous short examples, in general a score to each section, revision examples being set at the end of each chapter, and at the end of the book. This mode of presentation is fairly rare in France ; of the books mentioned above, only that of MM. Th. Leconte and R. Deltheil is constructed on this plan. In some books, even, which are reproductions of courses of lectures, sets of examples to be solved are completely

* [The Mathematical Association owes a debt of thanks to Professor Devisme for this article, written at the request of the Editor of the *Gazette*. It was thought that a comparison of teaching methods and ideals at about Scholarship stage would be both interesting and fruitful. For a concrete basis to the comparison, a modern and widely-used textbook was desirable, and the appearance of a third impression of *Advanced Trigonometry* (C. V. Durell and A. Robson) suggested that this book would exactly serve the purpose.]

absent. Another remark concerns the considerable number of examples—more than two thousand—of which the statements and the answers occupy more than half the 330 pages of the book, and this without counting some eighty examples solved in the course of the text, for the authors frequently prefer to elucidate the general case by starting from suitably chosen particular cases rather than by attacking it directly ; their aim is rather to encourage the student to think about each problem than to impose on him their own point of view. When the pupil has worked conscientiously through this book he will have not only an appreciable scientific equipment but a real dexterity and an excellent working method, which amounts to saying that the influence of the authors on the training of their young readers is only the more powerful because it is discreet.

The examples to be solved are stated very concisely, usually in one or two lines ; I have found only a score running to five or seven lines of text, one alone (p. 240) seeming to me to be stated verbosely. This contrasts strangely with examples set in French examinations, where the statement will fill one or two pages, a problem being a multiplicity of questions depending one on the other so that an initial error may vitiate the whole solution.

This diversity of points of view deserves some consideration. It cannot be denied that examples set to a student who is making his first acquaintance with a certain subject ought to contain only one difficulty to be overcome and ought to be only simple exercises in application. But the presence of longer questions is desirable because it is good to have several difficulties to be overcome at once, so as to learn to distinguish among these difficulties those which are essential and thus to learn to isolate these more conveniently. This is an excellent preparation for everyday life, even while it is dangerous to proceed too far in this direction and to train pupils whose skill in solving such questions is only evidence of a warped mind, dangerous as all premature specialisation is dangerous.

The remarks made from now onwards by the present writer bear on the questions discussed in the book ; I know of no equivalent book in France, the material being usually included in a " Cours d'Analyse". The advantage of this novel arrangement is that it allows innovations in method ; certain questions being isolated from those to which they are ordinarily attached take on thereby a certain independence. Unfortunately it will not be possible to make a final judgment on the success of this arrangement until the other books being prepared by the authors, *Advanced Algebra*, *Advanced Calculus* and *Companion Volume on Analysis*, shall have appeared. Reference to certain points in the course of the volume may appear to be too succinct, but the careful logic found in other parts is a sure guarantee that these small criticisms made on one volume will disappear when the whole series has appeared ; but I would have liked the authors to have indicated more precisely what line of development they count on giving to certain matters which are here seen from the more restrictive trigonometrical point of view.

At the moment we can say that as it is the book can be placed in the hands of French students in the classes of " Mathématiques Spéciales " preparing for the " Concours des Grandes Ecoles (Ecole Polytechnique, Ecole Normale Supérieure, Ecole Centrale, Ecoles des Mines, Ecole Navale, . . .) " as well as the students in the Faculties of Sciences. This shows that except for small differences of method and curriculum * the level and the material of French and English teaching is perfectly comparable. We might have doubted this *a priori*.

Now to turn the pages of the book and note suggestions arising from its perusal.

Chapter I is a very concise summary of the various properties of the triangle, being practically a collection of formulae, the reader being referred for proofs to Durell and Wright, *Elementary Trigonometry*, and Durell, *Modern Geometry*. After a number of triangles to solve and examples on the use of subsidiary angles, we find a revision of the formulae relating to the inscribed and escribed circles and the nine-points circle ; we may note here differences of notation, French authors using S and p in place of $\triangle$ and s to denote the area and semi-perimeter of a triangle. Formulae for medians and bisectors are obtained by employing centroids and the Apollonius theorem (which except for notation is that known to French teachers as Stewart's relation). Following this we find the classical relations between the principal points connected with a triangle ; the examples here are generally classical results. Then we have some notes on the solution of triangles defined by any set of its elements and an excellent half-page on the errors in the final result arising from errors in the data, with illustrative examples and examples for solution.

Chapter II gives in the same style a revision of the properties of the cyclic and circumscribable quadrilaterals, with some notes on the general quadrilateral. The number of results contained in the three and a half pages of text in this chapter brings into strong relief the concise style and economical reasoning of the authors.

Chapter III begins with the solution of simple trigonometrical equations, and the finding of an angle given one of its circular functions or an equation between the circular functions of the angle and of its multiples. In the solution of the example :

" *Find the number of real roots of the equation* $x = 3\pi(1 - \sin x)$ "

a graph is used ; I should have liked to see the use of graphs placed more in evidence, since the examples for solution following this are in fact modelled a little too obviously on the illustrative example.

After studying the bisection and trisection of arcs by means of the algebraic equation of degree three, the chapter ends with a very brief introduction (one page) to the inverse circular functions. We may observe the difference in notation, $\sin^{-1} x$ being used instead of

* Account must be taken of a difference in age of which we shall speak later. It should be noticed that the teaching for " Mathématiques Générales " given in the Faculties is more rapid than that for " Mathématiques Spéciales ", giving the mathematical ideas indispensable in the study of Physics.

arc sin x as in France, the latter notation having the advantage that the use of a capital letter serves to distinguish the principal value $y = \text{Arc sin}\, x\ (-\frac{1}{2}\pi \leqslant y \leqslant \frac{1}{2}\pi)$, the complete function being indicated by $y = \text{arc sin}\, x$.

Chapter IV studies the logarithmic and exponential functions. The logarithm function is introduced by considering the area bounded by a rectangular hyperbola, one of its asymptotes and two parallels to the other asymptote. By this process the authors obtain the classical properties of the function (growth, differential coefficient and addition formula). The examples for solution are interesting, some leading to approximative calculation of certain logarithms, others containing an analogous study of other functions of x, notably the function $y = \text{arc tg}\, x = \tan^{-1} x$. The exponential function is then obtained as the inverse of the logarithm ; considering the general properties of inverse functions, the usual results for the growth, differential coefficient and addition formula for the exponential are immediately derived. This theory completed, the applications to the integrals

$$\int \frac{dx}{1-x^2}, \quad \int \text{tg}\, x\, dx$$

are given or set as examples, and the chapter ends with a sequence of applications of the inequality

$$1 - \frac{1}{t} < \log t < t - 1,$$

such as finding the limits

$$\lim_{x \to 1} \frac{\log x}{x-1}, \quad \lim_{x \to \infty} \frac{\log x}{x}, \quad \lim_{x \to 0} \frac{e^x - 1}{x},$$

and proving the existence of Euler's constant.

Among the examples set as applications of this two-sided inequality, is the discussion of

$$\lim_{x \to 0} \frac{\log (1+x)}{x}$$

a result which plays a fundamental part in the corresponding section in the French textbooks and is investigated directly. The examples at the end of the chapter are fairly classical. We may note among the simpler examples an application of Napier's approximation formula

$$\log \frac{a}{b} \doteq \tfrac{1}{2}(a-b)\left(\frac{1}{a} + \frac{1}{b}\right), \quad \text{if } \frac{a}{b} - 1 \text{ is small.}$$

Some examples, classified as difficult, lead the student to discover for himself certain properties of the functions $\Gamma(n)$ and $B(m, n)$.

The method adopted in French books is in general quite different. They begin by defining a^x for rational x and then for irrational x by means of the idea of sections of rationals ; from this a study is made of the function $y = a^x$, where x is the independent variable. The

logarithm is introduced as the function inverse to this. To obtain its differential coefficient, the differential coefficient of the exponential function is first obtained by the following method :

$$y=a^x, \quad \Delta y=a^{x+\Delta x}-a^x=a^x(a^{\Delta x}-1);$$

now put $a^{\Delta x}-1=u$, whence $\Delta x=\log_a(1+u)$, which gives

$$\frac{\Delta y}{\Delta x}=a^x\frac{u}{\log_a(1+u)}\sim a^x\cdot\frac{1}{\log_a e}=a^x La.$$

In this the logarithm is made to play an essential part ; the sequence of ideas has always seemed to me to be somewhat unsatisfactory. Note that in the meantime it has been shown that

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

exists ; this limit is represented by the symbol e.

Exceptions to this very general mode of procedure are rare, and may now be noted. M. R. Garnier introduces the function e^x as the sum of the series

$$1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots,$$

shows that it satisfies the functional relation $f(x+y)=f(x)f(y)$, verifies that, for rational m, $f(m)$ can be identified with e^m, and generalises this to the case of irrational m. The logarithm is now defined as the inverse of this function.

Only MM. Th. Leconte and R. Deltheil introduce first the logarithm and then the exponential by the same means as in the book under discussion, a second instance of a very close similarity of ideas between the different authors.

Chapter V also gives rise to some useful reflections ; it deals with power series.* Here again the fact that the book is independent of other sections of Analysis enables it to utilise methods different from those followed in France. After a very short treatment of power series giving the definitions which are indispensable it passes on at once to the study of particular series. Doubtless, series ought to be dealt with in one of the projected volumes, but this cannot be stated definitely in the absence of any indication ; this presentation may appear artificial and the importance of the vital concepts is, to say the least, decidedly obscured.

After studying geometric progressions and certain limits, such as

$$\lim_{n\to\infty}x^n \text{ for } 0<x<1, \quad \lim_{n\to\infty}\frac{x^n}{n!},$$

the authors proceed to the expansions of $\sin x$ and $\cos x$ in series. To obtain these they make use of the fundamental result that for

* It is usual in France to call these " series entières ", which seems to me a defective terminology.

every interval $0<x<a$ in which $f(x)$ is integrable and positive, we have the inequality

$$\int_0^x f(t)\,dt>0.$$

Starting from the inequality $f(x)=1-\cos x>0$, they deduce by successive integrations the sequence of inequalities

$$1-\cos x>0,\quad x-\sin x>0,\quad \frac{x^2}{2}+\cos x-1>0,\quad \frac{x^3}{3!}+\sin x-x>0,$$

arriving finally at inequalities of the form

$$s_{2p}<\sin x<s_{2p+1}$$

with

$$s_{2p+1}-s_{2p}=\frac{x^{4p+1}}{(4p+1)!}\to 0,\quad \text{as } p\to\infty.$$

A complementary note provides this process with appropriate rigour.

The expansions of $\log(1+x)$ and $\tan^{-1}x$ are obtained by term-by-term integration of the expansions of $1/(1+x)$ and $1/(1+x^2)$; the convergence is established by a discussion of the magnitude of the remainder term. From this an evaluation of π is deduced, and reference is made to a more rapidly converging approximation obtained from Machin's formula. All this could find its place word for word in a French textbook.

The chapter ends with a discussion of the exponential series; the expansion is obtained by expressing the fact that the function defined by the series

$$a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots$$

is to satisfy the differential equation $y'=y$ with the initial condition $y(0)=1$, and we are very correctly reminded that the possibility of an expansion in series has not been demonstrated. After a method based on Maclaurin's formula is given for completeness, a third method is given using integration by parts, based on the existence of recurrence formulae for the integrals

$$u_n=\int_0^x e^{-t}t^n\,dt\quad (n \text{ a positive integer}).$$

It is easily deduced that

$$1+\frac{x}{1}+\frac{x^2}{2!}+\ldots+\frac{x^m}{m!}=e^x\left\{1-\frac{u_m}{m!}\right\}$$

and it only remains to show that $\lim_{m\to 0}(u_m/m!)=0$, which the authors proceed to do. The evaluation of e follows immediately and we are referred for further details to the *Companion Volume on Analysis*. The chapter closes with examples on the summation of series, some worked out and others for solution.

I have considered the line of thought in this chapter in some detail, since the spirit inspiring its construction is very different from that of French teaching although whole sections could be imported without difficulty into French teaching.

In the several treatises mentioned at the beginning of this article, the theory of numerical series is presupposed, and the method adopted comprises statements and proofs of the essential properties of power series; investigation of the interval of convergence, investigation of the remainder after the nth term, integration and differentiation term-by-term; then in the body of the text or as applications there are the usual expansions. Besides, since terminating expansions have already been obtained for almost all such functions during the study of differentiation and of Taylor's and Maclaurin's theorems, there remains, as I have said above, only the investigation of the remainders after the nth term to settle whether or not expansion in a non-terminating series is valid. The aim of Messrs. Durell and Robson, to provide simply the expansions of such functions as are useful in a book on Trigonometry, is quite different from this and is essentially practical; I should, however, very much like to see how they will link up this chapter with the corresponding chapters in the companion volumes.

Of Chapter VI it is only necessary to say, for the sake of completeness, that it deals with the hyperbolic functions; the text and the examples could without difficulty be incorporated in any French textbook.

We can pass rapidly over Chapter VII, which deals with the theory of projections and its applications, noting that questions involving the sense of a projection are very elegantly elucidated. The chapter concludes with summations of certain well-known series, such as

$$\sum_1^n \cos(\alpha+k\beta),\ \sum_1^n \sin(\alpha+k\beta),\ \sum_1^n u_k \text{ where } u_k=f(k+1)-f(k).$$

At the beginning of Chapter VIII, I observed this sentence on the subject of irrational numbers: "For many purposes it is enough to have a number which *approximately* satisfies an equation, and this is the reason why the introduction of irrationals is not urgent". This is to the point and expresses the authors' mode of thinking particularly well. The aim of their book is essentially practical; they are attempting to supply the student with the tools necessary to his work, avoiding any needless complications; and it is enough to signalise cases where delicate handling is required without entering into further detail.

Chapter VIII deals with the theory of complex numbers, introduced as pairs of real numbers arranged in a certain order $[a, b]$. Operations with these numbers are defined in this notation. The symbol i is brought in to facilitate the progress of the theory and represents the number [0, 1] and it is only very much later that the fact emerges that the operational technique of the numbers $[a, b]$ written in the form $a+ib$ comes to the same thing as operating algebraically on these numbers with the additional convention that $i^2=-1$. All this early part of the chapter is excellent, the principles of permanence of form and of extension of meaning being kept well in view. There is also an excellent twenty-line historical note on

the history of the introduction of complex numbers, and it may be regretted that the authors have not attempted to do something similar for the other chapters, where such indications are often fragmentary. This reproach may be addressed equally to French works, where the history of science is too much neglected, only one of the books mentioned, namely that of MM. G. Illiovici and A. Sainte-Lagüe, containing any attempt to improve this state of affairs. The uses of the Argand diagram bring the chapter to an end, with examples and results of well-known types. In passing, the difference in terminology between the two countries may be observed :

$$\begin{aligned} \text{Modulus} &\to \text{Module} ; \\ \text{Amplitude} &\to \text{Argument} ; \\ \text{Argument} &\to \text{Affixe} ; \\ \text{cis}\,\theta &\to 1_\theta . \end{aligned}$$

Chapter IX deals with De Moivre's theorem, which is proved for positive and negative, integral and fractional values of the index n. When n is fractional, the different values (we should say in France the different *determinations*) of $(\cos\theta + i\sin\theta)^{p/q}$ are considered, the cases in which p and q are or are not co-prime being distinguished ; the concept of the principal value is clearly defined. Next comes the investigation of $z^{p/q}$ and the graphical representation of the different values of $z^{p/q}$. As applications of De Moivre's theorem the authors give the formulae for expressing $\cos^n\theta$ and $\sin^n\theta$ as sums of sines and cosines of integral multiples of θ, as well as the multiplication formulae for $\sin n\theta$, $\cos n\theta$, $\tan n\theta$ in terms of $\sin\theta$, $\cos\theta$ and $\tan\theta$, and the determination of polynomials P_{n-1}, Q_n, such that

$$\sin n\theta = \sin\theta P_{n-1}(\cos\theta), \quad \cos n\theta = Q_n(\cos\theta).$$

The calculation of

$$C = \sum_0^{n-1} x^k \cos k\theta, \quad S = \sum_0^{n-1} x^k \sin k\theta$$

is made by evaluating $C + Si$.

The exposition of these two chapters is not radically different from that given in French books ; it should be observed that the programme for the class of " Mathématiques Spéciales " lays it down that De Moivre's theorem is not to be studied in the case when n is not an integer.

Chapter X considers uniform functions of a complex variable. It begins with matters relating to series whose terms are complex ; this time the authors indicate definitely that the groundwork of the theory of convergence will be found in the *Companion Volume on Analysis*. The summary of the principal properties to be used is clearly laid down, and we pass at once to the study of particular problems. First there is the geometric progression (why is there not a brief note showing that the function $1/(1-z)$ can be expanded as $1+z+z^2+\ldots+z^n+\ldots$ if $|z|<1$ and as

$$-\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots+\frac{1}{z^n}+\ldots\right) \text{ if } |z|>1 ?$$

I believe that a pupil only really grasps the meaning of the concept of the circle of convergence when he sees that it is related to the way in which an expansion in series of functions is made ; one or two simple examples are sufficient to focus attention on this fundamental fact, and textbooks are apt in general to fight rather shy of this point of view) ; then comes a study of the exponential series, of the addition formula for e^z, and of the expression of $\cos z$, $\sin z$ and $\tan z$ as sums of exponentials. The chapter closes with a comparison of the circular and hyperbolic functions, and there is no other point which merits particular attention.

The substance of Chapter XI, on roots of equations, is only to be found in French books in a scattered form or as examples ; in its exposition the authors demonstrate their originality. The chapter opens with the formation of equations whose roots are given, a problem evidently related to that of the symmetric functions of the roots of an algebraic equation. For example, suppose that an equation whose roots are $\cos 2\pi/7$, $\cos 4\pi/7$, $\cos 6\pi/7$ is to be found ; observing that the equation $\cos 4\theta = \cos 3\theta$ is satisfied by these values, we deduce an algebraic equation of the fourth degree in $c = \cos\theta$ which has these roots and the root $c = 1$, so that dividing by $c - 1$ we finally obtain the required equation

$$8c^3 + 4c^2 - 4c - 1 = 0 ;$$

from this, using the theory of equations, we can deduce the numerical value of expressions such as

$$\sec^2 \frac{2\pi}{7} + \sec^2 \frac{4\pi}{7} + \sec^2 \frac{6\pi}{7} .$$

There is a large selection of examples, which should assist a student to acquire that facility which is desirable in the handling of algebraic or trigonometrical manipulation. The idea of *essentially distinct* roots of a trigonometrical equation, by which is meant those roots which do not differ from another by a multiple of π, is put strongly in evidence. Here again the examples solved and to be solved could be of use to French readers.

Chapter XII, as a sort of inverse to the preceding chapter, discusses the decomposition of polynomials in one variable into linear factors. The chapter begins with the factorisation of $x^n - 1$, and the decomposition is made into linear factors and into real factors of the first or second degree. The factorisations of $\sin n\theta$ and

$$x^{2n} - 2x^n \cos n\alpha + 1,$$

which follow find no place in French teaching ; certain formulae in this section necessitate the introduction of infinite products, so the link with infinite series is pointed out and will doubtless be studied in more detail in another volume. After these few pages, whose pedagogical utility is not very apparent to the French reader, the authors deal with the decomposition of rational functions into partial fractions, and again the French reader must express his sur-

prise that the only case considered is that in which the denominator is the product of distinct linear factors. The question is thereby restricted. Will it be studied in another volume? There is no indication that it will, while the student using this book will be incapable of putting into partial fractions so simple a function as

$$\frac{1}{(x-1)^3(x-2)^2}.$$

Chapter XIII deals with multiform or many-valued functions of a complex variable; here is studied the function $\log w$ and its different values, and it is clearly shown that the set of values taken by $\log w^n$ is different from the set of values of $n \log w$. The logarithmic series is investigated, starting with the expansion

$$z - \frac{z^2}{2} + \frac{z^3}{3} \cdots (-)^{n-1}\frac{z^n}{n} + \cdots,$$

which is then separated into its real and imaginary parts, and by means of this example the authors bring in the concept of the circle of convergence. Then the functions z^n, z^w, sum function of the binomial series, $\log_z \zeta$ are considered and the chapter closes with some pages on the inverse circular and hyperbolic functions. All this is a little formal, and general principles are perhaps somewhat obscured. The concepts of transformations from one plane into another, of the *rôle* played by the different values and their possible interchanges as the independent variable describes a closed curve, these would give cohesion to a chapter which, as it stands, appears to be made up of material not suitably welded together; it might well seem artificial to a student meeting these topics for the first time. Let it be added that if the French books do not incur this reproach it is not because they deal with this point on the lines which I have suggested as desirable, but rather because the curriculum of the class of "Mathématiques Spéciales" is much less rich and contains in reference to functions of a complex variable only the following lines "Fonctions e^z, $\cos z$, $\sin z$, $\operatorname{sh} z$, $\operatorname{ch} z$ pour z complexe. Egalités

$$e^z e^{z'} = e^{z+z'}, \quad e^{x+iy} = e^x(\cos y + i \sin y)",$$

with some other restrictions. This programme is, of course, rather more than covered both in books and lectures.

Finally, there is by way of an appendix a chapter collecting together several fundamental results with sketches of methods for obtaining them. Here we meet with expressions which apparently are functions of an angle θ but which are in reality constant, such as

$$\cos^2\theta + \cos^2(a+\theta) - 2\cos a \cos\theta \cos(a+\theta),$$

identities connecting the circular functions of angles of a triangle, methods of eliminating one or more variables from trigonometric equations. To end the chapter there are numerous examples, and following this there are revision examples on the whole volume, arranged in ten sections of eight examples each.

Fifty pages of answers to the examples set during the course of the book (answers without comments) and an alphabetical index bring us to the end of the book. It is to be regretted that French books do not adopt the excellent practice of indexing alphabetically, only the book by MM. G. Illiovici and A. Sainte Lagüe of those mentioned above possessing such an index. In the other books, the student who has forgotten a definition is obliged to run over the preceding text again to find it by himself, and that is the cause of many disappointments.

We have thus arrived at the appropriate place for concluding remarks. From what has been said it would appear that the book under consideration, intended for boys of fifteen to eighteen, treats of rather more matters than are dealt with in books on corresponding subjects in French teaching, intended for boys of seventeen to twenty ; on the other hand, this teaching goes deeper, and assumptions are more closely analysed, a difference readily understandable, since it is intended for slightly older pupils with a more highly developed critical sense.

The assertion that of the books taken to afford comparison the treatise of MM. Th. Leconte and R. Deltheil is nearest in spirit and in actual presentation to the book of Durell and Robson is definitely symptomatic. The French book is in fact one of a collection intended to popularise knowledge, and the motto " Vulgariser sans abaisser " of the collection has always been followed scrupulously ; the aim of the collection is to provide for boys or for anyone interested in some problem a convenient working method and a clear approach to the point at issue. This tends to show that English teaching is essentially more practical and aims more immediately at activation than French teaching, where much more time is spent in the study of first principles and in the refinements of method before coming to applications.

Another remark is that of the books mentioned, those written by professors at the Universities or by authors more directly influenced by advanced teaching are more fairly comparable with the English book. This again is understandable ; the exchange of ideas between teachers in different countries is much more important for those in the Universities than for those in the Lycées or equivalent establishments. It may be observed that M. Th. Leconte, whose book has been mentioned, is one of the two Inspectors-general for the teaching of mathematics ; thus the guiding ideas in the two countries are comparable, as well as the teaching actually given.

Doubtless the English reader will be surprised at my having discussed the book of MM. Durell and Robson page by page ; the procedure may appear to them to be puerile, since this admirable book is widely known throughout England. But in order to study the manners and customs of a foreign country, we may make a voyage in order to see for ourselves, or we may rely on observations made on tourists from that country and their reactions to usages with which we are familiar. Not being able to present to my English

colleagues a view of all the French textbooks, I have attempted to play before them the less gratifying *rôle* of the tourist; by observing my reactions they will be able to learn something of the general ideas in the mind of a teacher of mathematics in France. The article has been prepared in the hope that it may be read by teachers of both countries, and that hope has prompted me to write with strict impartiality so that all may derive some instruction.

I trust that I have not too much misrepresented the main paths of exposition chosen by the various authors mentioned. It was with pleasure that I agreed to prepare this article at Mr. Broadbent's request, even though I knew it would be a somewhat laborious task; but I am sure that efforts in this direction can lead to a better reciprocation of knowledge between teachers in these two neighbouring countries, and I hope that an English colleague will be found to undertake the analogous task. Finally, may I offer thanks in advance to any readers who may be kind enough to send me comments and criticisms.

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France.

1116. A passion for physical science had spread widely through the nation. Except in the University of Leyden, where it was taught by an eminent professor named s'Gravesand, the great discovery of Newton had scarcely found an adherent on the Continent till it was popularised by Voltaire in 1728, but in England it had already acquired an ascendancy. Bentley, Whiston and Clarke enthusiastically adopted it. Gregory and Keill made it popular at Oxford, and Desaguliers, who gave lectures in London in 1713, says that he found the Newtonian philosophy generally received among persons of all ranks and professions, and even among the ladies, by the help of experiments.—W. E. H. Lecky, *History of England in the XVIIIth century*, I, 74-5 (Cabinet edition).

1117. The Schism Act . . . was intended to deprive them [the Dissenters] of the means of educating their children in their faith. . . . Some important clauses, however, were introduced by the Whig party qualifying its severity. They provided . . . that the Act should not extend to any person instructing youth in reading, writing, or arithmetic, or in any part of mathematics relating to navigation.—W. E. H. Lecky, *History of England in the XVIIIth century*, I, 118-20 (Cabinet edition).

1118. Upon the accession of the Hanoverian dynasty, however, Governmental encouragement of literature almost absolutely ceased. It is somewhat singular that the son of the Electress Sophia, who had been the devoted friend of Leibnitz, and the nephew of Elizabeth of Bavaria, who had been the most ardent disciple of Descartes, should have proved himself, beyond all other English sovereigns, indifferent to intellectual interests.—W. E. H. Lecky, *History of England in the XVIIIth century*, II, 80 (Cabinet edition).

WALLIS'S PRODUCT FOR $\pi/2$.

(From *Arithmetica Infinitorum*, J. Wallis, 1656.)

By George A. Dickinson.

From some viewpoints the beginnings of Analysis were marked by the publication, in 1656, by J. Wallis of the *Arithmetica Infinitorum*, in which he obtains the product $\frac{\pi}{2}=\frac{2.2.4.4.6.6}{1.3.3.5.5.7}$ *ad inf.*

He defines, very early in the book, a "limit" in this way (considering a series which tends to the value 1/3): "When the number of terms increases, the excess above 1/3 decreases, so that at length it becomes less than any assignable quantity; moreover, if the number of terms becomes infinite, the excess will have vanished completely". We still use substantially the same definition.

The following is an analysis of Wallis's investigation and methods by which he obtains the $\pi/2$ product. It is worthy of note that the Calculus had not been invented and very little was known about series. Many of his results are obtained by induction, but they are easily justifiable in the light of modern knowledge.

In one of his geometrical applications, however, Wallis is definitely in error; he gives the length of the Spiral of Archimedes, whose polar equation is $r=a\,.\,\theta$, between the limits $\theta=0$ and $\theta=\theta_1$, as half the length of the circular arc of radius $r_1=a\,.\,\theta_1$ subtending an angle θ_1 at the centre;

i.e. he gives arc $OSP=\frac{1}{2}\,.$ arc ACP

$$=\tfrac{1}{2}\,.\,a\,.\,\theta_1^2.$$

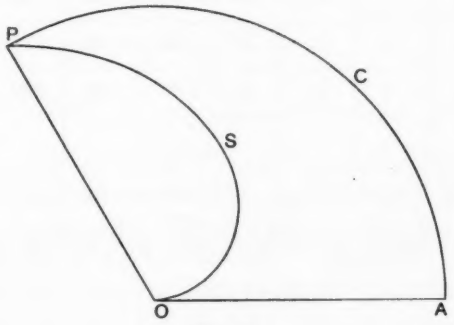


Fig. 1.

Actually, arc $OSP=a\int_0^{\theta_1}\sqrt{1+\theta^2}\,.\,d\theta$

$$=\frac{a}{2}\cdot\sinh^{-1}\theta_1+\frac{a\theta_1}{2}\,.\,\sqrt{1+\theta_1^2}.$$

Wallis investigates $\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} \frac{(N^p - k^p)^n}{N^{pn}} = \frac{1}{W(p,n)}$, say.

If we put $\frac{k}{N} = x$ and write the sum as an integral, we have

$$\frac{1}{W(p,n)} = \int_0^1 (1-x^p)^n \, . \, dx$$

$$= \frac{\Gamma(n+1) \, . \, \Gamma(1/p)}{p \, . \, \Gamma(n+1+1/p)} .$$

Now if n is a positive integer, this becomes

$$W(p,n) = \frac{(n \, . \, p+1)\overline{(n-1} \, . \, p+1) \ldots (p+1)}{n \, . \, p \, . \, \overline{n-1} \, . \, p \ldots p} .$$

Now Wallis expresses values of $W(p, n)$ in a table (for n a positive integer). This table is :

$p=$	$n=$ 0	1	2	3	4	5	
0	1	1	1	1	1	1	
$\frac{1}{1}$	1	2	3	4	5	6	
$\frac{1}{2}$	1	3	6	10	15	21	
$\frac{1}{3}$	1	4	10	20	35	56	
$\frac{1}{4}$	1	5	15	35	70	126	
$\frac{1}{5}$	1	6	21	56	126	252	

Note that $W(p, n)$ increases as n increases for fixed p ; and also

$$\frac{W(p,n+1)}{W(p,n)} = \frac{\Gamma(n+2+1/p)}{\Gamma(n+2)} \cdot \frac{\Gamma(n+1)}{\Gamma(n+1+1/p)}$$

$$= \frac{n+1+1/p}{n+1}$$

$$= 1 + \frac{1}{p \, . \, (n+1)} .$$

But Wallis is interested in finding the value of $W(2, \frac{1}{2})$; for he considers the area of a quadrant of a circle of radius R to be made up of A lines of which s_r is typical.

Suppose the distances between consecutive lines are all equal to a, so that $A \,.\, a = R$.

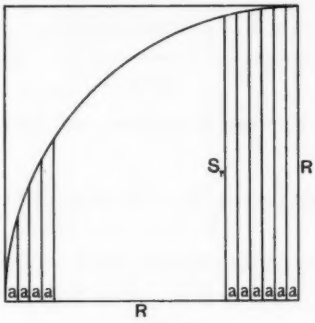


FIG. 2.

Then, as A tends to infinity, $\sum_{r=0}^{A} s_r$ will tend to the area of the quadrant.

But $$s_r^2 = (R + r \,.\, a) \,.\, (R - r \,.\, a).$$

So $$\text{area of quadrant} = \lim_{A \to \infty} \sum_{r=0}^{A} (R^2 - r^2 \,.\, a^2)^{\frac{1}{2}}.$$

And the ratio, $$\frac{\text{area of quadrant}}{\text{area of square on radius}} = \lim_{A \to \infty} \frac{1}{A} \sum_{r=0}^{A} \frac{(R^2 - r^2 \,.\, a^2)^{\frac{1}{2}}}{R},$$

which is clearly $1/W(2, \frac{1}{2})$, and actually $= \pi/4$.

Incidentally,
$$\begin{aligned} W(2, \tfrac{1}{2}) &= \frac{2 \,.\, \Gamma(\frac{3}{2} + \frac{1}{2})}{\Gamma(\frac{3}{2}) \,.\, \Gamma(\frac{1}{2})} \\ &= \frac{2 \,.\, \Gamma(2)}{\frac{1}{2}[\Gamma(\frac{1}{2})]^2} \\ &= 4/\pi \\ &= \square, \text{ say.} \end{aligned}$$

Wallis now assumes that $\dfrac{W(p, n+1)}{W(p, n)} = 1 + \dfrac{1}{p \,.\, (n+1)}$ is true for non-integral n.

Then, since $W(2, 0) = 1$ and $W(2, \frac{1}{2}) = \square$, we obtain this series:

$n = -\frac{1}{2}$; 0; $\frac{1}{2}$; 1; $3/2$; 2; $5/2$; 3; $7/2$, etc.;

$W(2, n) = \frac{1}{2}\square$; 1; $\square$;

$$\frac{3}{2}; \quad \frac{4}{3}\square; \quad \frac{3 \,.\, 5}{2 \,.\, 4}; \quad \frac{4 \,.\, 6}{3 \,.\, 5}\square; \quad \frac{3 \,.\, 5 \,.\, 7}{2 \,.\, 4 \,.\, 6}; \quad \frac{4 \,.\, 6 \,.\, 8}{3 \,.\, 5 \,.\, 7}\square, \quad \text{etc.};$$

say, $= \alpha$; a; β; b; γ; c; δ; d; ϵ, etc.;

He then assumes tacitly that $\frac{a}{\alpha}, \frac{\beta}{a}, \frac{b}{\beta}, \frac{\gamma}{b}, \frac{c}{\gamma}, \frac{\delta}{c}$, etc., is a decreasing sequence;

i.e. he assumes $\dfrac{W\left(2, \frac{n+1}{2}\right)}{W\left(2, \frac{1}{2}n\right)} > \dfrac{W\left(2, \frac{n+2}{2}\right)}{W\left(2, \frac{n+1}{2}\right)}$, where n is integral.

As a matter of fact this is true, but it is far from obvious. We have to show that

$$\left\{\int_0^1 (1-x^2)^{\frac{n+1}{2}} . dx\right\}^2 < \left\{\int_0^1 (1-x^2)^{\frac{n}{2}} . dx\right\} \times \left\{\int_0^1 (1-x^2)^{\frac{n+2}{2}} . dx\right\}.$$

Now Holder's inequality, namely, $\Sigma a . b < (\Sigma a^k)^{\frac{1}{k}} . (\Sigma b^{k'})^{\frac{1}{k'}}$, where $k' = \dfrac{k}{k-1}$ and $k > 1$, has its analogue for Stieltjes' integrals

$$\int u . v . d\phi < \left(\int u^k . d\phi\right)^{\frac{1}{k}} . \left(\int v^{k'} . d\phi\right)^{\frac{1}{k'}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

with the same conditions for k and k', and where $\phi(x)$ is an increasing function.

Consider $\displaystyle\int_0^1 (1-x^2)^{\frac{n+1}{2}} . dx.$

Take $u = 1$; $v = (1-x^2)^{\frac{1}{2}}$; $\phi = \displaystyle\int (1-x^2)^{\frac{n}{2}} . dx$, giving the L.H.S. of (1).

Then $\left\{\displaystyle\int_0^1 (1-x^2)^{\frac{n}{2}} . dx\right\}^{\frac{1}{2}} . \left\{\int_0^1 (1-x^2)^{\frac{n+2}{2}} . dx\right\}^{\frac{1}{2}}$

is the R.H.S. of (1), for $k = 2 = k'$. So Wallis's assumption is justified.

Since, then, $\frac{a}{\alpha}, \frac{\beta}{a}, \frac{b}{\beta}, \ldots \frac{l}{\lambda}, \frac{\mu}{l}, \frac{m}{\mu}, \ldots$ is a decreasing sequence,

$$\frac{\mu}{\lambda} = \frac{l}{\lambda} \times \frac{\mu}{l} > \left(\frac{\mu}{l}\right)^2 \text{ and } \frac{m}{l} = \frac{\mu}{l} \times \frac{m}{\mu} < \left(\frac{\mu}{l}\right)^2.$$

Thus $$\sqrt{\frac{m}{l}} < \frac{\mu}{l} < \sqrt{\frac{\mu}{\lambda}}.$$

Consider the sequences

$$a;\quad b\ ;\quad c\ ;\quad d\ , \text{ etc.},$$

$$1;\ \frac{1.3}{2};\ \frac{1.3.5}{2.4};\ \frac{1.3.5.7}{2.4.6}, \text{ etc.};$$

and

$$\alpha;\quad \beta\ ;\quad \gamma\ ;\quad \delta\ ;\ \text{etc.},$$

$$\tfrac{1}{2}\square;\ \tfrac{1}{2}\square \cdot \frac{2}{1};\ \tfrac{1}{2}\square \cdot \frac{2.4}{1.3};\ \tfrac{1}{2}\square \cdot \frac{2.4.6}{1.3.5}, \text{ etc.}$$

It is clear that $\frac{\mu}{\lambda}$ and $\frac{m}{l}$ both tend to the value unity as the number of terms preceding them in the sequence tends to become infinite; in which case $\frac{\mu}{l}$ is equal to unity:

i.e. $$1=\frac{\frac{1}{2}\square\times\frac{2}{1}\times\frac{4}{3}\times\frac{6}{5}\times\ldots\frac{2r}{(2r-1)}\times\ldots \textit{ad inf.}}{1\times\frac{3}{2}\times\frac{5}{4}\times\frac{7}{6}\times\ldots\frac{(2r-1)}{2r}\times\ldots \textit{ad inf.}},$$

giving $$\tfrac{1}{2}\square=\frac{1.3.3.5.5.7.7.9\ldots}{2.2.4.4.6.6.8.8\ldots}\ \textit{ad inf.}$$

hence $$\frac{\pi}{2}=\frac{2.2.4.4.6.6.8.8\ldots}{1.3.3.5.5.7.7.9\ldots}\ \textit{ad inf.}$$

More precisely, Wallis states that

"$$\frac{3.3.5.5.7.7.9.\ 9.11.11.13.13}{2.4.4.6.6.8.8.10.10.12.12.14}\times\sqrt{1\tfrac{1}{14}}$$

$$<\square<\frac{3.3.5.5.7.7.9.\ 9.11.11.13.13}{2.4.4.6.6.8.8.10.10.12.12.14}\times\sqrt{1\tfrac{1}{13}},$$

and so on, as far as you like".

To sum up, Wallis makes two assumptions:

(1) $\dfrac{W(p,n+1)}{W(p,n)}=1+\dfrac{1}{p(n+1)}$ (non-integral n),

(2) $\dfrac{a}{\alpha}, \dfrac{\beta}{a}, \dfrac{b}{\beta}, \ldots$ is a decreasing sequence.

This second assumption is very mysterious. In the first place Wallis gives no reason for thinking that it is true (and it is difficult to see why he should have thought it was); in the second place he had no need to make it at all, for there is a very much simpler way of completing his argument.

He could have used the fact that $\alpha, a, \beta, b, \gamma, c, \ldots$ is an increasing sequence such that m/l and μ/λ tend to the value unity as the number of terms tends to become infinite. So that $l<\mu<m$, and when the number of terms is infinite $l=\mu=m$:

i.e. $1\times\frac{3}{2}\times\frac{5}{4}\times\frac{7}{6}\times\ldots$ *ad inf.* $=\frac{1}{2}\square\times\frac{2}{1}\times\frac{4}{3}\times\frac{6}{5}\times\ldots$ *ad inf.*,

giving immediately, since $\square=\dfrac{4}{\pi}$,

$$\frac{\pi}{2}=\frac{2.2.4.4.6.6.8.8\ldots}{1.3.3.5.5.7.7.9\ldots}\ \textit{ad inf.}$$

G. A. D.

"A WEIGHTY MATTER."

By G. H. Grattan-Guinness.

The eternal question of the distinction to be drawn, in teaching elementary Mechanics to youngsters, between mass and weight has recently thrust itself upon me in discussing with a Science colleague the choice of a textbook on General Science suitable for pupils, aged about 11 or 12 years, beginning the subject. The first year General Science syllabus which he has drawn up excludes as far as possible all quantitative work; my own Mathematics department is left to deal with the fundamental teaching of all work involving units, not only of mere length, area, and such things as invariably occur in elementary mathematical teaching, but also of such others as temperature and our old friends mass, force, weight, density, and so on, which will in time be used in the Mechanics course for which the Mathematics department assumes responsibility and to which the Science department will refer incidentally as it feels the need.

Acting on the principle that, although a chosen textbook may exclude certain topics which one would prefer it to contain and yet not be deemed therefore unacceptable, one must nevertheless at least avoid the presentation of a topic in a definitely erroneous manner, I found no difficulty in persuading my colleague against certain books.

Almost all the Science books inspected included work on density, which he is excluding from his syllabus, but which is in mine. I therefore first examined the various treatments of density. Of about a dozen different books on "Elementary Science", more or less "general", fully half boldly defined density as "weight per unit volume"; one modified this by a reasonable exposition on mass and then made a fresh start; in not a single case was mass dealt with first, and density given its proper definition once and for all. In another connection, I was recently choosing between two almost equally good books on Mechanics for School Certificate purposes; my choice was soon settled when I found that the author of one of them was measuring forces in tons, pressures in pounds per square inch, and so on.

During the last decade or so, a vast output of work on elementary Mechanics has manifested itself, and a universally acknowledged improvement is taking place in the presentation of the subject to young pupils. Thus, it comes nowadays as a shock to find any gravitational unit of force which does not include the word "weight" as part of its name, or to discover any confusion in the use of the terms "force" and "pressure" that was once not uncommon; and, in particular, I was overjoyed recently to discover in one textbook (Siddons, Snell and Dockeray: *Elementary Mechanics*) the official introduction of the unit "foot-pound weight" for the measurement of work, as I have for some time been insisting that my pupils shall use instead of the misleading "foot-pound".

It is all the more amazing, therefore, to find that, even among the most modern publications, some authors of textbooks are guilty of such slipshod statements as I have mentioned above ; is it any wonder that the wretched pupil, using such books, finds the " mass-weight " question one of almost superhuman knottiness ?

The Association's *Report of the Teaching of Mechanics* tried, over six years ago, to sort the matter out, and undoubtedly has given great assistance in that direction. However, a careful survey of that part of the report which deals with these questions (§ 6, pages 20-30 ; especially §§ 6.2, 6.4, and 6.6) leads one to the conclusion that the Committee were unwilling to make any definite recommendations : it would perhaps have been more profitable to have " laid down the law " and thus forced the problem into open discussion. As it is, the most valuable aspect, to my mind, of this section of the *Report* is in the form of a problem to the teacher : " When are we suggesting good methods ? when are we pointing out errors ? are we always consistent ? " It is obvious that certain authors, even after six years, have not profited by a personal solution of this problem, which perhaps arises mainly from the fact that the *Report* is printed in one type only, no distinction, either by paragraphing or by change of fount, being made to distinguish between statements of erroneous practice and suggestions for improved method.

I cull the following extracts from the *Report* as needing, to my mind, careful consideration before acceptance ; in some cases, indeed, I contend that they should be vehemently denied :

(1) " The word ' pound ' is used in mechanics in at least two senses, if not by pure mathematicians certainly by engineers. This may be regrettable, but we cannot change it." (§ 6.2, p. 21.)

(2) " In statics we measure forces in pounds." (§ 6.2, p. 21.)

(3) ". . . long-established custom sanctions the use of ' pound ' as a unit of force. . . . " (§ 6.2, p. 22.)

(4) " The mass of a body is measured by the number of standard pounds which it balances in the scales ; so is its weight." (§ 6.4, p. 24.)

(5) " . . . a body of mass *m* lb. has a weight of *m* lb.-wt., or more briefly has a weight of *m* lb." (§ 6.4., p. 26.)

(6) " The mass of a body then is defined as the number of pounds, tons, grams, or kilograms which it balances in the scales, *i.e.* that which is commonly, but less precisely, styled its weight." (§ 6.6, p. 28.)

(7) " . . . we may still speak of tying a weight on to the end of a string. . . ." (§ 6.2, p. 22.)

(8) " . . . showing how it [mass] can be measured scientifically without the use of the balance." (§ 6.4, p. 25.)

The laying of the blame on the engineers is, I think, justified, and,

after them, I would cite, in the main, the physicists. It seems that these two bodies of men are in too many cases concerned with little but "getting the answer", no matter by what process. But are we, as teachers of *Mathematics*, the science of logical consistency, to bow down to the misapplications of these so-called practical men who are, after all, novitiates in our order? Is it not our duty to make every endeavour to lead them to the strait gate along the narrow path rather than follow them along the broad highway of construction "from first unprinciples"? I suggest therefore that the Mathematical Association solemnly recant its despairing admission, "We cannot change it", and substitute a determination to set its mechanical house in order. For, when all is said and done, we *are* changing it: I have already pointed out that it is rare to find a modern textbook following in the wake of (2) and (3); and, in any case, surely the last thing we should do is to plead the excuse of "long-established custom"—what would have become of the A.I.G.T. if that slogan had been allowed prominence?

I contend that the last phrases of (4), (5), and (6), and the whole of (7), should be regarded as matters of evil practice, and should be rigorously excluded from our teaching. In (6), we might better say: "... commonly, but *quite inaccurately*, styled its weight"; while, if any definition of weight has been given in terms of earth-pull, (7) is fantastic. As regards (8), I see no reason why, in an elementary course (that is, before school specialisation sets in, unhappily at only too early a date in many cases), any effort should be made to divorce the idea of mass from the balance. A whole-hearted association of these two gives the youngster an easy conception of mass *on which his whole course of work can be consistently built up to include all the mechanical principles that he will require*, without the necessity of any possible *volte face* such as the *Report* envisages. (§ 6.2, p. 22.) At the same time, I should be the last to deny the value of a complete revision of ideas in which mass is treated by an alternative method apart from the balance—especially, *e.g.*, on the lines of § 6.4 (*c*), pp. 25, 26—but only to the more advanced pupils.

I append a brief outline of the method I have used for some time in teaching fundamental units to children as young as 11 and 12 years, not with any idea that it is perfect, but rather as a basis for future discussion which may eventually (dare we hope?) get the whole affair settled once and for all. It should be noted that this is not intended to represent a highly concentrated dose to be taken in as short a time as possible. The sequence below merely indicates the order in which the various stages of the work appear; no one stage necessarily follows *immediately* on the preceding, nor is there any compulsion to complete one stage in one lesson or consecutive lessons.

(1) The first idea of units in measurement—length. Standard

units, multiples and submultiples—tables. Extension to area and volume, with capacity. Similarity and distinction between volume and capacity.

N.B.—British and/or metric units: both must be dealt with at some time or other: it is left to the teacher to determine whether both shall be taken simultaneously or which shall precede the other. So in all that follows.

(2.1) *Matter and mass*: "Anything that occupies space is called 'matter'"—a short discourse emphasising the universality of matter, *e.g.* its various states, etc.—"When we want to measure the quantity of matter in a body, we say that we are measuring its mass. *This mass has nothing to do with the volume of the body* —that measures the space occupied by the matter, not the matter itself. How then shall we measure the mass of a body?" (Don't tell them—yet!)

(2.2) Take two identically constructed objects and a uniform rule balanced about its middle; obtain the experimental fact that the rule remains balanced when the objects are placed at equal distances on either side of the suspension—the idea of the ordinary scales or balance. "The two objects have been made identical, as far as we can see; they must therefore have equal masses (whatever we mean by that), for, being identically equal, they are equal in *every* respect. Thus, equal masses will balance on either side of an ordinary pair of scales, and we shall use this fact to measure the mass of a body, by finding what *known* mass (see 2.3) it will balance. In other words, what we usually call 'weighing' we shall now call 'measuring mass'".

(2.3) Units of mass—definition of standard (pound; gram or kilogram)—explanation of how one could get multiples and submultiples from it—tables. Point out that these tables (16 oz. = 1 lb., etc.) have probably been taught as "tables of weight", avoirdupois and otherwise; they will now receive their proper title of "tables of mass". "Understand clearly that what the butcher, the baker, and even perhaps the candlestick-maker, call the 'weight' of an object, meaning the measurement obtained with a pair of scales, is not its weight at all (I shall tell you later on—see 3.3—what we really mean by the 'weight' of a body); to be scientifically exact you must call this the measurement of the body's *mass*. Of course, in everyday life you must be sensible enough to conform to general practice in this matter: if you are buying bacon, do not say to the shopman, 'I'll take that piece; just find its mass for me, please'. This term 'mass' is a *scientific* term; in everyday life you will probably talk about 'weight'—wrongly, mind you, but in order to keep the peace; when you continue your studies in science, however, you *must* be exact, for science is nothing if not exact, and to be exact you *must*, if you are thinking about measuring a body with a balance, refer to that body's *mass*, not its weight."

N.B.—Too much time and trouble cannot be devoted to this part of the work. It is essential that the pupil be made quite clear not only about scientific theory, as it were, but also about the discrepancy between this and popular practice.

(3.1) *Force*—only a very elementary idea necessary at this stage —" anything in the form of a pull or a push ". The more formal definition will come after Newton's Laws of Motion have been taken.

(3.2) Newton's Law of Gravitation—a short talk on its general meaning and the universality of its application. The mathematical form of its statement may be too difficult at this stage, and general terms will suffice : " The bigger the mass of the attracting body, the bigger the force ; the farther away, the smaller the force ".

(3.3) The idea that everything (*e.g.* a particular pupil) is acted upon by myriads of forces—from other pupils, from the teacher, from the objects in the room, from the building, from the whole earth, from astronomical bodies ; of these, only one force—the earth's—is of numerical importance—explain why the others are negligible (see 3.2). Thus, each body is subject to an earth-pull, which is called its " weight ". Emphasise that the weight of a body is a force, and has nothing to do with quantity of matter ; hence, *weight is not measured on a balance.*

(3.4) Hooke's Law verified experimentally with a spiral spring, using standard masses to produce extensions, and explaining why the force causing the extension must be proportional to the mass hung, since, by Newton's Law of Gravitation, equal masses suffer equal attractions from the same body, and hence have equal weights from the same earth—and so *pro rata* ; construction and use of the dynamometer.

N.B.—It is difficult to avoid mentioning the common term " spring balance ", but, to avoid confusion in the pupil's mind, it is preferable to be consistent in the use of the term " dynamometer ". " *The balance for measuring mass, the dynamometer for weight.*"

(3.5) Units of weight—definition of standard : " the weight of a pound mass " (etc.), abbreviated to " a pound's weight ", and thus " 1 lb. wt."—tables, similar to, but not identical with, the familiar tables of mass (see 2.3), the word weight being added throughout : 16 ozs. wt. = 1 lb. wt., etc.

(4.1) *The weight of a body is variable*—depends on the position of the body with respect to the earth ; on the moon, for example, is the moon's pull, for the earth's pull is negligible, just as, on the earth, the moon's pull is negligible. An imaginary journey to the moon, seated on the hook of a flying dynamometer—variations in scale readings—all highly fanciful, but most useful in getting the facts home, including the region of no weight, that is, of no impression from the hook !

(4.2) *But the mass of a body is invariable*: the argument that, wherever the body is, it is the same piece of matter; and/or another moon journey (the return?), seated on one side of a balance, counterpoised by sufficient *so-called* "weights"—any alterations in earth-pull on one's own mass must be counteracted by corresponding alterations to the counterpoising mass (if for no other reason, because there is no possible reason to suppose these alterations unequal), and the registration of one's own mass is thus unaffected, even in the region of no weight.

N.B.—These differences also need much emphasis: they are of fundamental importance—see (5) below. The idea of no change in registration of mass will be made clearer when "moments" are understood.

(5) *True and false units.* "The unit '1 lb.' of mass is an absolute unit, a universal standard, but the unit '1 lb. wt.' is not a true standard. The standard lump which we call 'the pound mass' has a weight (that is, an earth-pull) in, say, Greenwich, which is called '1 lb. wt.', and in, say, Washington it has a weight, substantially different from the former, which is still called—by the Americans, be it noted—'1 lb. wt.'. The difference between the two is of only about 1 part in 1000, but it is nevertheless a difference. So far, then, there is no real standard of force (we shall meet that later on), all units such as 1 lb. wt., 1 ton wt., 1 gr. wt., 1 kg. wt., etc., varying slightly from place to place on the earth's surface, but not sufficiently to cause any *practical* difficulty—and we are not concerned with astromechanics."

N.B.—This matter will clarify itself later on when it is realised that 1 lb. wt. $=g$ pdls. exactly (and so on), that the poundal (etc.) is fixed, and that g varies from place to place. If meanwhile a child asks how is it possible to measure differences in weight, if the unit of measurement varies from place to place, point out that each dynamometer is (in theory) calibrated for a particular region; that a dynamometer showing "1 lb. wt." in the neighbourhood of Greenwich would register something quite different on the moon, and, if sufficiently accurate, something different again in Washington; in each case we should have to use a different instrument with a different spring in order to obtain a reading of "1 lb. wt." in each of the new localities. *Measured in terms of the Greenwich unit*, the weight shows itself to be variable; measured in terms of a variable unit, it may appear constant!

(6) Summarising:

(*a*) We use a balance for measuring mass, a dynamometer for measuring weight.

(*b*) Mass and weight are quantities quite different in nature.

(*c*) The mass of a body is constant and is measured in a fixed system of units; its weight is different in both respects.

Yet mass and weight have this one great feature in common: *in any given region*, at least as far as the earth is concerned, the standard pound mass has a weight which is known as the (local) pound weight, and so a body whose mass is m lb. has a (local) weight of m lb. wt.—the same *numerical* quantity of units in both cases. Since, therefore, in every problem involving experimental measurement and/or computation, we assume, unless directed otherwise, that our problem deals only with our own locality, it is chiefly this *number* "m" that is required, irrespective of units, and it is immaterial whether we weigh a body (*i.e.* determine its mass) on a balance (as we ought) or on a dynamometer (for quickness)—we shall get the same number "m" in any case, within the limits of the accuracy of the instruments. A similar but reversed argument applies to weight. The important thing is this: "Bear in mind that, in using the dynamometer, you are actually finding that the *weight* of the body is $7\frac{1}{2}$ (say) *lb. wt.*, and, if you really want its mass, you must omit the last word 'weight'; so, with the balance, you are actually measuring the *mass* as $7\frac{1}{2}$ *lb.*, and, if you really require the weight, you must now add the word 'weight' to the unit".

(7) *Measurement of forces generally* will be made in terms of "lb. wt.", etc., as is that of weight itself; for weight is only a particular type of force, and, having chosen a set of units for this case, we shall obviously find it convenient to continue with the same for all forces.

That completes a somewhat lengthy, but far from wasted, introduction, developed, I admit, on a big and bold assumption of the Law of Gravitation; and when it is all once clearly fixed in the pupil's mind—and, though it may take some fixing, I have found it not beyond the mental capacities of the average 11- or 12-year-old—the way is open for the development of mechanical principles along rightly defined paths. Density, for example, when required, whether early or late in the course, comes in quite accurately as "mass per unit volume", and every other new measure can also receive a logically correct unit. Thus, until the poundal, dyne, and their derivatives appear on the scene, every mechanical quantity involving force can—and should—bear as part of its name the word "weight". I personally insist on my pupils measuring their forces in "lb. wt." (etc.), work and energy in "ft.-lb. wt.", power in "ft.-lb. wt./sec.", moments in "lb. wt.-ft.", impulse and momentum in "sec.-lb. wt.", pressure in "lb. wt./sq. ft.", and so on; and I have found that, except for an occasional slip, they have used such terms consistently and thus obviously with understanding. I find, too, that the question of dimensions and the equating thereof is much simplified if all the units are stated according to a

consistent scheme: once a child has overcome the initial difficulty of realising why the dimensions of force are $[MLT^{-2}]$ (*i.e.* why the equation "$P=mf$" has not only a numerical but also a dimensional meaning), he will quite easily pick up for himself the thumb-rule that the inclusion of the word "weight" in a unit is the equivalent of an added $[LT^{-2}]$ to the other, more obvious, dimensions of that unit, but the omission of that vital word is likely to cause endless confusion. At the same time, moreover, he is far less likely to make errors due solely to a superfluous or an absent "g".

This brings me to my last point. This problem of "g or not g?" which so often confronts a young pupil arises partly from the sins of commission on which I have enlarged—loose statements of units—and partly from sins of omission—the practice of complete absence of any units. It is a debatable point which is the cause of more difficulty in elementary Mechanics. In my view, it is no cure of the former ill to adopt the latter process. The young pupil who meets, say, an elementary statical problem involving "tension T" and "weight W", and later on encounters a dynamical exercise including "tension T" and "weight mg", is bound to have difficulties, and a teacher working on these lines is surely courting disaster. Why should g appear in one case and not in the other? In my own practice, I never allow youngsters in any circumstances to refer to "a mass m", "a force P", "a distance s", "a velocity v", and so on; and even with more advanced pupils I tend to discourage it. Thus, with beginners, I insist on every unit being written down exactly every time, line after line; this takes longer, but it does get the exact principles instilled. With older people, I still make the same demand in all cases of numerical exercises; with regard to the algebraic type of problem, where no units are given, I suggest either the invention of units—say, in the C.G.S. system—or the addition of such legends as "abs. units" or "grav. units", as the case may be, to the various symbolical expressions. I admit that with the really brilliant pupil this is perhaps not absolutely essential—he is sufficiently good to be allowed to dispense with Queensberry Rules, as it were; but with his intellectually less favoured brethren I submit that it is an invaluable safeguard—the extra time involved in the extra writing is a form of insurance against errors.

But there is another, less utilitarian, more fundamental, reason why our quantities should not suffer anonymity. Of recent years there has grown up, in the teaching of Algebra, a strong and healthy doctrine that symbols stand for numbers, and numbers only. "x is a *number*; the length of the carpet cannot be x; you must say 'x feet', 'x metres', or whatever units you are using." This precept is urged in every modern textbook on Algebra that I have met, and rightly so—the purpose is part of that training in right thinking that all teachers of Mathematics strive to give their pupils. Yet, let what is mathematically an identical problem be met with in a Science lesson, and principles can seemingly take wings!

Chemists with their "atomic weight 16" and "solubility 45", physicists with their "focal length f" and "resistance R", engineers (and I include teachers of Mechanics) with their "work done mgh" and "acceleration g", are all contributing to play havoc with the exactness of scientific thought that they are supposed to be trying to inculcate in their pupils.

The result of such teaching methods is seen only too frequently. It is not unknown to find examinees urged to "take g as 981 cm. per sec. per sec."—an obvious contradiction in fundamentals. A certain well-known but rather old-fashioned textbook on elementary Mechanics is full of problems worded on this plan: "A particle starts with velocity u; shew ... that $u=8{\cdot}165$ feet per second"—a perfect example of mathematical topsy-turveydom. Are we to encourage our pupils to adopt such a practice as this, which, even if we deny it as absolute error, we must at least admit to be "bad form"?

I do urge that the time is over-ripe for a definite drive towards uniformity in the principles underlying our teaching, whether it be of the so-called "Pure" Mathematics or any of its various "Applied" forms. If we teach one doctrine in our Algebra lessons, we must surely "practise what we preach" when we come to apply that Algebra. Otherwise we are guilty of what is, after all, mathematical immorality—a serious charge to sustain. Reform, if it be brought about, should emanate from the Mathematics people rather than their Science colleagues, for the same reasons as I have put forward before; could not the Mathematical Association itself make suitable deliberation, and issue a proclamation?

1119. Each new discovery disclosed the wide range and uniformity of law, and the theory of gravitation proving that its empire extended over the most distant planet had a mental influence which can hardly be overrated. From this time astrology, witchcraft, and modern miracles, which a few generations before presented no difficulty to the mind, began silently to vanish, not so much in consequence of any controversy or investigation, as because they no longer appeared probable, no longer harmonised with the prevailing conception of the government of the world.—W. E. H. Lecky, *History of England in the XVIIIth century*, III, 4 (Cabinet edition).

1120. But for the most part divines in England cordially accepted the great discoveries of their time, and freethinkers appear to have had no suspicion that men of science were their natural allies. When Collins ascribed the decay of witchcraft to the growth of freethinking, Bentley retorted that it was not due to freethinkers, but to the Royal Society and to the scientific conception of the universe which that society had spread. Nearly all the early members of the Royal Society, nearly all the first teachers of the Newtonian philosophy, were ardent believers in revelation.—W. E. H. Lecky, *History of England in the XVIIIth century*, III, 6-7 (Cabinet edition).

1121. Unity may be defined as a ratio of equality: zero and infinity as the two terms in the ratio of greatest possible inequality.—R. L. Stevenson. *Memories and Portraits*, Tusitala edition, p. 74. [Per Mr. C. E. Kemp.]

MATHEMATICAL FILMS.

THE Film Sub-Committee has had an opportunity of viewing some recent films and submits these descriptions of them, together with a few notes on their probable value in teaching, for the convenience of members of the Association who may be interested.

SIMPLE MACHINES (Kodak)

This film gives a number of interesting pictures of machines in everyday activities : levers (a see-saw, a crane, a man's arm, scales), wheels (a windlass, use of pulleys in lifting weights, block and tackle), inclined planes (a motor-car on a hill, a spiral stairway, an auger boring through wood, an aeroplane propeller, a motor-boat propeller), and combinations of these machines (a steam-shovel loading a truck).

The various parts of the film would be of real value in connection with the teaching of the different machines, if only they could be used separately ; but their value from this point of view is much diminished by the inclusion of all of them in one film. As it stands, it could be used only as a general introduction to a course on simple machines, or as a general review at the end of the course, or both.

THE ANGLE-SUM OF A TRIANGLE (Mr. B. G. D. Salt)

In this film, by means of movements of lines and angles, (1) the fact of the equality of the angle-sum to two right angles is shown by fitting the three angles together, (2) the " proof by revolution ", and (3) the usual proof depending on the parallel theorems, are shown.

To use this film in teaching the theorem would evidently be to substitute a visual demonstration for the exercise in reasoning that the proof ought to afford the pupil ; but it would furnish a fresh and vivid means of revision. It is an interesting, rather than a really useful, film.

THE EQUATION $\ddot{X}+X=0$ (Mr. R. Fairthorne and Mr. B. G. D. Salt)

The authors of this film explain that it is to be regarded as a working model. It is, in fact, a set of moving diagrams rather than a complete film with a definite beginning, middle, and end.

The basic element of the diagram is a disc, of variable radius, rotating at constant speed. From the perimeter of this disc unrolls a string, whose length is therefore equal to the integral of the radius with respect to the time. The principle is the same as that of wheel and disc planimeter, of which the diagram is a simple form.

Two of these discs are shown. The radius of one represents the acceleration, so the length of string unrolled represents the velocity. This string determines the radius of a second disc, which is therefore proportional to the velocity. The string unrolled from the velocity disc represents the distance, and is arranged to determine the radius of the first (acceleration) disc so as to satisfy the equation. At all instants the radii of the two discs will satisfy the equation, so the motion of the diagram represents the solution.

The diagram and its motion make immediately evident :

(1) The meaning of the relations symbolized by the mathematical equation.
(2) The symmetry of the relations. The velocity and the distance may be interchanged with only an alteration of phase.
(3) The inevitability of periodic roots. If the second term had a negative coefficient, that is, if the discs turned in opposite directions, the discs would increase or decrease in size indefinitely.

The next sequence superimposes upon the diagram the loci of the ends of the rotating radii. These are double circles whose diameters equal the maximum radius, and whose axes of symmetry are at right angles. This brings out :

(4) The quarter period difference in phase between the velocity and the distance.
(5) The identity of the roots with the sine and cosine.

The last sequence shows the harmonic curves that result if the velocity and distance are plotted on a time base.

The fact that the initial diameters of the two discs are arbitrary shows that the equation can be made to fit any two conditions laid down beforehand, although the type of motion is unchanged.

It can be shown, by the transformation $T=Nt$, that the solution of the equation $\ddot{X}+N^2X=0$ is represented by the same film run N times as fast.

Throughout the film a white disc represents a negative quantity, a black disc a positive.

This film should prove of real value to a VIth Form beginning the study of differential equations. It affords, what it would otherwise be impossible to import into the class-room, a visible illustration of the relations involved in these equations. It is most ingeniously devised, well carried out and very interesting to watch.

THE DIFFERENCE BETWEEN SIMPLE SLIP AND CREEP IN A BELT RUNNING ON A PULLEY (Dance-Kaufmann. No. 9a1)

This film gives a simple illustration of the difference between the effects known as slip and creep in belts.

A part of it shows simple slip in which the whole of the belt is slipping with reference to the driving pulley. Under these conditions the belt would quickly fall off the pulley unless special means were taken to keep it on.

In the other part the elasticity of the belt is very much exaggerated and the belt is shown running on to the pulley in tension. Just before the belt runs off on the slack side the tension is relieved and the belt contracts to its unstressed dimensions ; in doing so it creeps along the face of the pulley and causes a loss of energy.

The film does not show the shear effect through the thickness of the belt.

HYPOCYCLOID GEAR (Dance-Kaufmann. No. 9b1)

This film may be used to illustrate either the mathematical treatment of the hypocycloid or to illustrate the mechanism in which use

is made of the property of the hypocycloid when the ratio of the diameters of the circles is 2 : 1.

In this particular case the trace is a straight line. This property was used in the early steam engines in which a 2 : 1 gear was employed to convert the straight line motion of the piston into rotary motion of the shaft.

The film may be run in either direction or reversed from left to right or from top to bottom.

INTERMITTENT MOVEMENT 8 : 1 (Dance-Kaufmann. No. 9b2)

This film illustrates a simple example of a movement which enables one shaft to drive another with a mean ratio of 8 : 1, but the movement of the driven shaft is intermittent while the movement of the driver is uniform. The film is intended to be used in place of a model. It should be run in the direction consistent with the faster shaft being the driver ; subject to this, the picture may be reversed from left to right if required.

These three Dance-Kaufmann films should prove useful in teaching the mathematics of engineering and may be recommended for that purpose.

ADDRESSES OF PRODUCERS, PRICES, ETC.

Simple Machines. Can be obtained from Kodak, Ltd., Kodak House, Kingsway, London, W.C. 2. Price £9 10s., or can be hired at 3s. per day.

Angle-Sum of a Triangle. Can be obtained from Mr. B. G. D. Salt, 5 Carlingford Road, Hampstead, London, N.W. 3. Price 30s.

Equation $\ddot{X} + X = 0$. Can be obtained from Mr. R. Fairthorne, Kirk Michael, Hillfield Road, Farnborough, Hants. Price 30s.

Dance-Kaufmann Films. Can be obtained from Dance-Kaufmann, 18 Upper Stanhope Street, Liverpool, either in loop form (a closed band which enables the sequence to be repeated indefinitely) or in the ordinary reel form. Prices: No. 9a 1, 12s. 6d., the other two 10s. each.

Note. The Dance-Kaufmann films are no longer distributed by Messrs. Steuart. In most cases the distribution is suspended for the next few months ; it will then be resumed from the Liverpool address. But the three films here described are already ready for distribution at that address.

All the films described above can be supplied in the 16mm. size.

It may be useful to call the attention of those interested in the use of films in schools to a leaflet entitled *Non-Theatrical Cinematograph Apparatus and Films*, recently issued (price 6d.) by the British Film Institute, 4 Great Russell Street, London, W.C. 1. It contains valuable information about projectors and films, the firms that supply them, and libraries from which films can be hired.

MARGARET PUNNETT.
On behalf of the Film Sub-Committee.

THE PILLORY.

IN the paper on *Statics and Dynamics* set in December, 1936, for entrance scholarships at Trinity, King's, Magdalene, Trinity Hall, Pembroke, Corpus Christi, Clare and Downing colleges, no less than four of the ten questions are open to some criticism.

No. 3 contains a mistake of sign in the answer. Such mistakes are perhaps inevitable. Attention is called to it only because it does not stand alone. It strengthens the presumption that insufficient consideration was given to this paper before it was set.

No. 9, a question about the simple harmonic motion of a spring, contains an answer which is reached only by supposing that the modulus λ has an unusual meaning : $T=\lambda x$ instead of $T=\lambda x/c$.

The mature mathematician will easily find out in No. 9 that λ has been used instead of λ/c, and he will see that the answer given in No. 3 is manifestly impossible. It is not fair to expect a schoolboy to do this. Nor is there any reason to expect that the good candidate will waste less time (looking for mistakes in his own work) than the other candidates.

Nos. 2, 8 are worth quoting in full. The first of them is an artificial problem belonging to the Victorian era. Heavy men do not go up weightless ladders in modern mathematics.

2. A long ladder of negligible weight rests with one end on the ground and the other projecting over the top of a wall of height h, the vertical plane through the ladder being at right angles to the wall. The coefficients of friction at the ground and the wall are μ_1 and μ_2 and the ladder makes an angle θ with the horizontal. Show that if a man tries to ascend the ladder, equilibrium will be broken by the ladder tilting or sliding according as $\mu_2 >$ or $< \tan\theta$. If equilibrium is broken by sliding, show that the man will be unable to reach the top of the wall unless $\mu_1 > \cot\theta$, and find the maximum height he can attain.

Show further that whatever may be the values of the frictions the man cannot ascend a vertical distance greater than $h\sec^2\theta$.

8. A train of mass M is moving with velocity V when it begins to pick up water at a uniform rate. The power is constant and equal to H. If, after time t, a mass m of water has been picked up, find the velocity and show that the loss in energy is

$$m(Ht+MV^2)/\{2(m+M)\}.$$

No. 8 might almost have been set in the form :

. . . the loss of energy is so and so, find the meaning of the question.

The answer to the riddle is as follows : the train is coasting at speed V ; it begins to pick up water and does so at a constant rate of k lb. per sec. ; to mitigate the effect a constant power H is switched on ; no allowance is to be made for work done in raising the water, although the answer implies that m and M are comparable ; also loss of energy means energy dissipated, not change in energy of the travelling body.

It would be interesting to know what arrangements are made by railway companies to enable trains to pick up water at a uniform rate when they are moving with variable velocities. Such puzzles as arise from these questions 2 and 8 may be good exercises for teachers, but they are not fair to the examinees.

CORRECTION.

Note 1220. *Sur le triangle isoscèle.*

La dernière phrase du paragraphe 1 doit être remplacée par la suivante :

Les triangles BME et CFN sont donc isoscèles ; de plus, ils sont équiangles, à cause des égalités d'angles

$$MEB = MAB = CAN = CFN.$$

P. 53, remplacer l'égalité

$$EBM = FCB$$

par

$$CBE = FCB.$$

V. THÉBAULT.

NON-ASSOCIATE POWERS AND A FUNCTIONAL EQUATION : POSTSCRIPT.

I find that the results given in my article in the *Mathematical Gazette*, February 1937, pp. 36-39, were obtained with greater generality and rigour by J. H. M. Wedderburn, *Annals of Mathematics*, (2) *24* (1922), pp. 121-140, to whom I offer apologies.

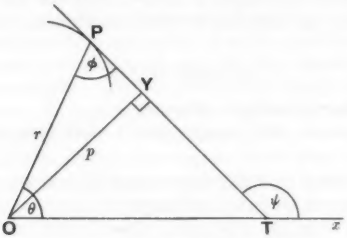
I. M. H. ETHERINGTON.

1122. South, it is true, and some other divines had denounced the Royal Society as irreligious; and Leibnitz afterwards attacked, on theological grounds, the Newtonian theory of gravitation, though he consoled himself, in one of his letters, by the reflection that it might furnish an argument for the Lutheran doctrine of consubstantiation. [Footnote] He writes to the Princess of Wales : "Mr. Newton prétend qu'un corps attire l'autre à quelque distance que ce soit, et qu'un grain de sable chez nous exerce une force attractive jusques sur le soleil sans aucun milieu ni moyen. Après cela comment sectateurs voudront-ils nier que par la toute-puissance de Dieu nous pouvons avoir participation du corps et du sang de Jésus-Christ sans aucun empêchement des distances ? C'est un bon moyen de les embarrasser—des gens qui par un esprit d'animosité contre la Maison d'Hanovre, s'émancipent maintenant plus que jamais de parler contre nostre religion de la Confession d'Augsburg, comme si notre Réalité Eucharistique étoit absurde ", Kemble's *State Papers*, p. 529.—W. E. H. Lecky, *History of England in the XVIIIth century*, III, 6 (Cabinet edition).

1123. Leibniz had been instrumental in securing the succession, and he abounded in expositions of constitutional policy. He professed himself so good a Whig that he attributed to that cause his unpopularity with many people in England, especially at Cambridge, and most of all at Trinity. He seems not to have known that his rival, Newton, was as good a Whig as himself, and indeed a much better one.—Lord Acton, *Lectures on Modern History*, p. 267-8.

MATHEMATICAL NOTES.

1233. *A proof of the formula* $\rho = r\,dr/dp$.



$$\psi = \theta + \phi.$$

Thus
$$\kappa = \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds}. \qquad \text{.....(i)}$$

Now
$$\frac{d\theta}{ds} = \frac{1}{r}\left(r\frac{d\theta}{ds}\right) = \frac{\sin\phi}{r}$$

and
$$\frac{d\phi}{ds} = \frac{d\phi}{dr}\cdot\frac{dr}{ds} = \frac{d\phi}{dr}\cdot\cos\phi.$$

Hence from (i)
$$\kappa = \frac{\sin\phi}{r} + \frac{d\phi}{dr}\cos\phi$$
$$= \frac{1}{r}\left(\sin\phi + \frac{d\phi}{dr}\,.\,r\cos\phi\right)$$
$$= \frac{1}{r}\frac{d}{dr}(r\sin\phi) = \frac{1}{r}\frac{dp}{dr},$$

and
$$\rho = r\,dr/dp.$$

F. J. TONGUE.

1234. *Feuerbach's Theorem.*

There is a familiar proof of Feuerbach's theorem by an inversion of which the centre is A', the midpoint of BC, and the radius is such that the incircle and the first excircle are self-inverse. Since the nine-points circle passes through A', its inverse is a line, and it is shown that this line is a common tangent to the incircle and excircle. If the line of centres AII_1 cuts BC in S, (AS, II_1) is harmonic, and it follows by projection on BC that S is the inverse of D, the foot of the altitude AH, and therefrom that the inverse of the nine-points circle passes through S. Up to this point there is no room for variation in the details; this note was begun in order to draw attention to a short proof of the remaining step, namely, that the inverse is necessarily parallel to the reflection of BC.

The inverse of the nine-points circle is perpendicular to the diameter through the centre of inversion A'. But the point diametri-

cally opposite to A' on the circle is P, the midpoint of AH, and $A'P$ is parallel to OA. The conclusion is immediate: since AO is the reflection in AS of AH, the line through S perpendicular to AO is the reflection in AS of the line through S perpendicular to AH, that is, of BC.

Although the steps that have been taken were designed to lead to a proof by inversion, we may well ask—and this is the second object of this note—whether, when so much is known, the appeal to inversion is not almost absurdly superfluous. Should we not use the established facts as the basis of the particular deduction we need, rather than refer to general theorems on the inversion of circles and on the preservation of tangency?

Let X be the point at which the incircle touches BC, and let ST be the second tangent from S to the incircle. Then the results available are (*i*) $A'S \cdot A'D = A'X^2$, (*ii*) ST is perpendicular to $A'P$. Let $A'T$ cut the incircle again in K, and let $A'P$ cut ST in V.

Firstly,

$$A'T \cdot A'K = A'X^2 = A'S \cdot A'D = A'V \cdot A'P,$$

since the angles at D and V are right angles. Hence KP is perpendicular to KT, that is, to KA', and K is on the circle of which $A'P$ is a diameter, namely, the nine-points circle.

Secondly, if U is the point diametrically opposite to T on the incircle, KP, being perpendicular to KT, passes through U. But the diameters TU, $A'P$, of the incircle and the nine-points circle, are parallel, since both are perpendicular to ST. Hence the point K, in which the lines AT, PU intersect, is on the line joining the midpoints of the diameters. That is, the line joining the centres of the circles passes through a point common to the circles.

H. N. HASKELL.

[This Note has been compiled from a number of separate letters I have received from Mr. Haskell: I have modified the details slightly to avoid "angle-chasing" and its ambiguities, but the outlines and the emphasis are his entirely.

Although the proof dispenses with the theory of inversion, it is not disputed that the proof would not have been devised without the guidance of that theory. This however is a boomerang argument, for there is no reason to suppose that the particular inversion itself would have been investigated unless the result to which it was leading had been known beforehand. E. H. N.]

1235. *Some remarks on the coefficient of restitution.*

The coefficient of restitution e which occurs in the discussion of the impact of elastic bodies can be defined in two ways:

(i) by Newton's experimental law of impact;

(ii) by the theory (due to Poisson) which defines e as the ratio of the impulse of restitution to the impulse of compression.

It is the purpose of this note to examine each of these definitions and to set out some of the assumptions implied in them. As a framework on which to hang the discussion, I take two examples.

Example (*a*). A body of mass m_1 moving with velocity u_1 impinges directly on a body of mass m_2 moving with velocity u_2. Find the resulting velocities.

Example (*b*). A sphere of mass m_1 is suspended from a fixed point by an inextensible string ; a second small sphere of mass m_2 moving with velocity V downwards in a direction making an angle α with the vertical impinges directly on the first sphere. Find the velocities of the spheres after impact.

For the sake of completeness let me now quote Newton's experimental law : " When two bodies A and B impinge, the velocity after impact of A, relative to B, is equal to $-e$ times the velocity before impact of A relative to B ; the velocities being measured along the common normal." Here it is tacitly assumed that e is independent of the velocities before impact and the shapes and masses of the bodies. When e is defined by (ii) the same assumptions are made ; there is, however, one important difference.

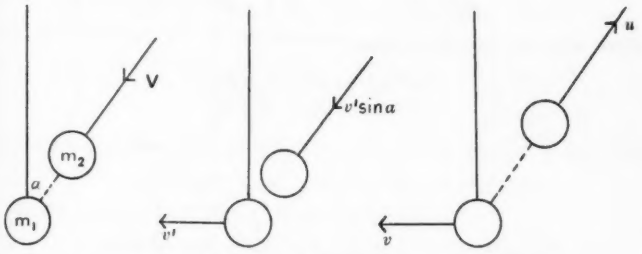
In definition (i) we assume that the only impulses acting on the bodies are the mutual impulses exerted between them. This is not done when defining e from (ii). Problems in impact can be divided into two classes, those in which the mutual impulses are the only ones acting on the bodies at the moment of impact, and those in which other impulses may act in addition to the mutual impulses. Example (*a*) is typical of the first class, example (*b*) of the second ; the impulsive tension in the string is of course the additional impulse.

Let us consider further the contention made in the last paragraph. If in defining e from (i) we do not assume that the mutual actions between the bodies are the only impulses acting, then e will depend on the other impulses, and for the impact of two bodies of given materials e will depend on the conditions of the particular problem discussed. Thus suppose that in examples (*a*) and (*b*) the spheres of mass m_1 are made of steel and those of mass m_2 are made of glass, then there appears to be no valid reason for supposing that the coefficients of restitution in the two examples are equal (as defined by Newton's law). We cannot say that the coefficient of restitution between steel and glass is, say, $\frac{3}{4}$, but only that for the particular problem discussed the coefficient of restitution between steel and glass is $\frac{3}{4}$. Therefore since e is to constant only for mass, velocity, shape, etc., but is to vary with the problem for given materials, there is little use in its introduction. The value of any physical experimental constant lies in the fact that once found it can be used to forecast the results of given sets of conditions. If e is to be of value, then it must depend solely on the materials of the impinging bodies and not on the class of the problem.

I claim, then, that in defining e according to (i) it is assumed that the only impulses acting on the bodies are the mutual impulses between them. The conclusion is that if we adopt method (i) we are justified in using it to solve problems of type (*a*) but not of type (*b*). It should here be mentioned that if we confine our attention to

problems of type (*a*), then taking (i) as our definition we can establish (ii) and *vice versa*. This is proved in any elementary textbook. In this class of problem neither definition has therefore any merit over the other.

Examples of type (*b*) I have shown can only be solved logically by defining *e* as in (ii). It is this point which has prompted this note. Most intermediate textbooks, and certainly most schoolbooks, offend in this respect. The curious point, however, is that the erroneous application of Newton's law gives the correct result in most of the cases which the writer has examined. I give below the solution of example (*b*).



Before impact. End of period of compression. After impact.

At the end of the period of compression let v' be the velocity of m_1; v' is of course horizontal. Since the relative velocity of the spheres is now zero, the velocity of m_2 is $v' \sin \alpha$ as shown.

Let v and u be the velocities of m_1 and m_2 after impact.

Let I be the impulse of compression; then eI is the impulse of restitution. Considering m_2 we have

$$I = m_2(V - v' \sin \alpha), \quad \text{.....(i)}$$

$$eI = m_2(u + v' \sin \alpha). \quad \text{.....(ii)}$$

Considering m_1 and resolving horizontally to avoid the impulsive tension in the string, we have

$$I \sin \alpha = m_1 v', \quad \text{.....(iii)}$$

$$eI \sin \alpha = m_1(v - v'). \quad \text{.....(iv)}$$

We have four equations to determine the four unknowns I, v', u and v, which give

$$u = V\{em_1 - m_2 \sin^2\alpha\}/(m_1 + m_2 \sin^2\alpha),$$

$$v = V\{(1+e)m_2 \sin \alpha\}/(m_1 + m_2 \sin^2\alpha).$$

In a textbook the solution is given thus:

The momentum equation in the horizontal direction is

$$m_2 V \sin \alpha = m_1 v - m_2 u \sin \alpha; \quad \text{.....(v)}$$

applying Newton's law along the line of centres,

$$u+v\sin\alpha=eV. \quad \text{.....(vi)}$$

Solving these equations we have u and v as above.

The question arises as to whether a given problem is determinate or not. This the author would like to discuss later.

W. H. OSBORNE.

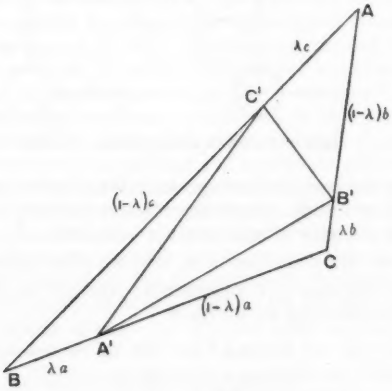
1236. *A property of the triangle.*

If on the sides BC, CA, AB of any triangle, we take points A', B', C', such that

$$BA' : A'C = CB' : B'A = AC' : C'B,$$

then, with the usual notation,

$$\frac{a'^2+b'^2+c'^2}{a^2+b^2+c^2}=\frac{\triangle A'B'C'}{\triangle ABC}.$$



Let $\quad AC'=\lambda c, \quad BA'=\lambda a, \quad CB'=\lambda b.$

Then
$$a'^2=\lambda^2c^2+(1-\lambda)^2b^2-2\lambda(1-\lambda)bc\cos A,$$
$$b'^2=\lambda^2a^2+(1-\lambda)^2c^2-2\lambda(1-\lambda)ca\cos B,$$
$$c'^2=\lambda^2b^2+(1-\lambda)^2a^2-2\lambda(1-\lambda)ab\cos C.$$

Adding, and replacing $bc\cos A$ by $b^2+c^2-a^2$, etc., we get

$$a'^2+b'^2+c'^2=(3\lambda^2-3\lambda+1)(a^2+b^2+c^2). \quad \text{.....(i)}$$

Again,

$$\triangle A'B'C' \equiv \triangle ABC-(\triangle A'B'C+\triangle B'C'A+\triangle C'A'B).$$

Thus

$$\frac{\triangle A'B'C'}{\triangle ABC}=\frac{\frac{1}{2}ab\sin C-\frac{1}{2}\lambda(1-\lambda)\{ab\sin C+bc\sin A+ca\sin B\}}{\frac{1}{2}ab\sin C}.$$

Since $a : \sin A = b : \sin B = c : \sin C$, we have

$$\triangle A'B'C' : \triangle ABC = 1 - 3\lambda + 3\lambda^2. \quad \text{.....(ii)}$$

Comparing (i) and (ii) we have the result. R. H. BLOMFIELD.

1237. *A theorem on three conics derived from a conic and a triangle.* Let S be any conic in the plane of the triangle XYZ, and let S_2, S_3 denote the conics $S + 2\phi zx$, $S + 2\psi xy$, where ϕ, ψ are any two constants. If (x_2, y_2, z_2), (x_3, y_3, z_3) are the poles of the line $x=0$ for the conics S_2, S_3, then

$$hx_2 + by_2 + fz_2 = 0, \qquad (h+\psi)x_3 + by_3 + fz_3 = 0,$$
$$(g+\phi)x_2 + fy_2 + cz_2 = 0\ ; \qquad gx_3 + fy_3 + cz_3 = 0\ ;$$

and since each of the determinants

$$\begin{vmatrix} h & g+\phi & \phi h+\psi g+\phi\psi \\ b & f & \phi b+\psi f \\ f & c & \phi f+\psi c \end{vmatrix} \quad \begin{vmatrix} h+\psi & g & \phi h+\psi g+\phi\psi \\ b & f & \phi b+\psi f \\ f & c & \phi f+\psi c \end{vmatrix}$$

vanishes, the line joining these two poles is the line (u, v, w), where

$$u = \phi h + \psi g + \phi\psi, \quad v = \phi b + \psi f, \quad w = \phi f + \psi c.$$

The line equations of the conics S_2, S_3 are

$$\Sigma_2 \equiv \Sigma + \phi M - \phi^2 m^2 = 0, \quad \Sigma_3 \equiv \Sigma + \psi N - \psi^2 n^2 = 0,$$

where Σ is the quadratic form reciprocal to S, and

$$M = -2gm^2 + 2hmn - 2bnl + 2flm,$$
$$N = -2hn^2 + 2gmn + 2fnl - 2clm.$$

From the values of u, v, w, we have

$$\begin{aligned}
\Sigma^{(1)} &\equiv Au + Hv + Gw = \phi\psi A, \\
\Sigma^{(2)} &\equiv Hu + Bv + Fw = \phi\psi H + \phi\Delta, \\
\Sigma^{(3)} &\equiv Gu + Fv + Cw = \phi\psi G + \psi\Delta, \\
M^{(1)} &\equiv fv - bw = -\psi A, \\
M^{(2)} &\equiv fu - 2gv + hw = \phi\psi f + 2\phi G - \psi H, \\
M^{(3)} &\equiv -bu + hv = -\phi\psi b + \psi G, \\
N^{(1)} &\equiv -cv + fw = -\phi A, \\
N^{(2)} &\equiv -cu + gw = -\phi\psi c + \phi H, \\
N^{(3)} &\equiv fu + gv - 2hw = \phi\psi f - \phi G + 2\psi H.
\end{aligned}$$

Hence,

$$\begin{aligned}
\Sigma^{(1)} + \phi M^{(1)} &= 0, & \Sigma^{(1)} + \psi N^{(1)} &= 0, \\
\Sigma^{(2)} + \phi M^{(2)} - \phi^2 v &= \phi\Phi, & \Sigma^{(2)} + \psi N^{(2)} &= \phi\Psi, \\
\Sigma^{(3)} + \phi M^{(3)} &= \psi\Phi, & \Sigma^{(3)} + \psi N^{(3)} - \psi^2 w &= \psi\Psi,
\end{aligned}$$

where $\Phi = \Delta + 2\phi G - \phi^2 b, \quad \Psi = \Delta + 2\psi H - \psi^2 c.$

It follows that with respect to the conic whose line equation is

$$\Sigma_2/\Phi = \Sigma_3/\Psi,$$

the line (u, v, w) is conjugate to every line in the plane; in other words, this conic is a pair of points lying in the line (u, v, w). That is to say, a pair of points of intersection of common tangents to the conics S_2 and S_3 lies in the line joining the poles of $x=0$ for these conics, and the equation of this pair of points is $\Sigma_2/\Phi=\Sigma_3/\Psi$.

If now we introduce a third conic $S_1 \equiv S+2\theta yz$, one pair of points of intersection of common tangents to S_3 and S_1 is $\Sigma_3/\Psi=\Sigma_1/\Theta$, and one pair of points of intersection of common tangents to S_1 and S_2 is $\Sigma_1/\Theta=\Sigma_2/\Phi$, where $\Theta=\Delta+2\theta F-\theta^2 a$. Obviously the three pairs of points of intersection whose equations have been found belong to a line-pencil of conics, that is, are the pairs of vertices of a quadrilateral.

If any conic cuts the sides of a triangle in V_1, W_1; W_2, U_2; U_3, V_3, and if S_1, S_2, S_3 are any three conics through W_2, U_2, U_3, V_3, through U_3, V_3, V_1, W_1, and through V_1, W_1, W_2, U_2, then there are one pair of points of intersection of common tangents to S_2 and S_3, one pair of points of intersection of common tangents to S_3 and S_1, and one pair of points of intersection of common tangents to S_1 and S_2, which are the three pairs of vertices of a quadrilateral.

In the reciprocal of this theorem, tangents must first be drawn from the vertices of a triangle to an arbitrary conic; if this conic degenerates to the circular points, the theorem becomes the theorem on focus-sharing conics which was the subject of a paper in the *Gazette* last July (Vol. XX, p. 182). E. H. N.

1238. *Note on focus sharing conics.*

In the July issue of the *Gazette*, Professor Neville proved that *If three ellipses are such that each pair has one common focus and two real points of intersection, the three chords of visible intersection are concurrent.*

In view of the time between the statement of the theorem and the publication of a proof another method may be of interest.

If r_1, r_2, r_3 denote distances from the foci A, B, C, the equations of the ellipses α, β, γ may be written

$$r_2+r_3=\lambda, \quad r_3+r_1=\mu, \quad r_1+r_2=\nu \text{ respectively.} \quad \ldots\ldots\ldots\text{(i)}$$

We may assume that γ has the greatest major axis,

$$i.e.\ \lambda'=\nu-\mu>0, \quad \mu'=\nu-\lambda>0.$$

Then the intersections of β, γ are the same as those of β and α'—the hyperbola (one branch)

$$r_2-r_3=\lambda'. \quad \ldots\ldots\ldots\text{(ii)}$$

Similarly the common points of α, γ are the intersections of α with β', the hyperbola

$$r_1-r_3=\mu'. \quad \ldots\ldots\ldots\text{(iii)}$$

We now have four conics $\alpha, \alpha', \beta, \beta'$ with a common focus C and it is natural to reciprocate for C.

We note first, however, that at the intersections of α, α'

$$r_3=\tfrac{1}{2}(\lambda-\lambda')=\tfrac{1}{2}(\lambda+\mu-\nu)=\delta \text{ say,}$$

(δ is positive because $\lambda>a>\lambda'$).

Again, at the intersections of β, β',

$$r_3=\tfrac{1}{2}(\mu-\mu')=\delta \text{ also.}$$

Hence a circle S, centre C, radius δ, passes through the common points of α, α' and β, β'.

Using this as the base circle for reciprocation we obtain four other circles $\bar{\alpha}$, $\bar{\alpha}'$, $\bar{\beta}$, $\bar{\beta}'$.

Now since α, α', S have two common points, $\bar{\alpha}$, $\bar{\alpha}'$, S have two common tangents meeting in L their centre of similitude.

Similarly $\bar{\beta}$, $\bar{\beta}'$, S have two common tangents meeting at their centre of similitude M.

But the external centres of similitude of three circles are collinear.

Hence LM is the line of collinearity for the centres of similitude of

(1) $\bar{\alpha}$, $\bar{\alpha}'$; (2) $\bar{\beta}$, $\bar{\beta}'$; (3) $\bar{\alpha}$, $\bar{\beta}'$; (4) $\bar{\alpha}'$, $\bar{\beta}$; (5) $\bar{\alpha}'$, $\bar{\beta}'$; (6) $\bar{\alpha}$, $\bar{\beta}$.

Reciprocally the pole of LM for S is the point of concurrence of the common chords of

(1) α, α'; (2) β, β'; (3) α, β'; (4) α', β; (5) α', β'; (6) α, β.

In particular from (3), (4), (6), we see that the common chords of α and γ, β and γ, α and β are concurrent.

[An alternative procedure is to use the focal distance formulae "$a \pm ex$" for an ellipse, "$ex \pm a$" for a hyperbola, to get the equations of the common chords. It follows that the point of concurrence is at distances $\lambda\lambda'/2a$, $\mu\mu'/2b$ from the perpendicular bisectors of BC, CA.

In particular, if $\lambda=\mu=\nu$, $\lambda'=\mu'=0$ (the ellipses have equal major axes), then the point of concurrence is the circumcentre of the triangle ABC.] B. E. LAWRENCE.

1239. *A necessary and sufficient condition for the convergence of certain series.*

I. If Σa_n is a series of positive terms in which the ratio of successive terms, can be expressed in the form

$$\frac{a_n}{a_{n+1}}=1+\frac{\mu}{n}+O\left(\frac{1}{n^\lambda}\right) \quad \text{(where } \lambda>1\text{)}, \quad \ldots\ldots\ldots\ldots\ldots(1)$$

the condition $a_n=o\left(\frac{1}{n}\right)$ is necessary and sufficient for the convergence of Σa_n.

Bromwich, who mentions this result in his *Infinite Series* (second edition, § 12.2, p. 41), suggests a method of proving it which involves the consideration of infinite products. The alternative proof given below would, however, appear to be simpler and more direct.

The condition (1) is equivalent to the following:

$$a_n \sim n^{-\mu}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

For, ν being sufficiently large (say $\nu \geqslant m$), we have from (1),

$$\log a_\nu - \log a_{\nu+1} = \frac{\mu}{\nu} + \frac{k_\nu}{\nu^\lambda} + \frac{k_\nu'}{\nu^2},$$

where k_ν, k_ν' are bounded as $\nu \to \infty$.

Summing for $\nu = m, m+1, \ldots, n$ and adding

$$\mu\left[1 + \frac{1}{2} + \ldots + \frac{1}{m-1} - \log(n+1)\right]$$

to both sides of the resulting equality,

$$\mu\left[1 + \frac{1}{2} + \ldots + \frac{1}{m-1}\right] + \log a_m - \log a_{n+1} - \mu \log(n+1)$$
$$= \mu\left[1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log(n+1)\right] + \sum_{\nu=m}^{n} \frac{k_\nu}{\nu^\lambda} + \sum_{\nu=m}^{n} \frac{k_\nu'}{\nu^2}.$$

Let $n \to \infty$ in the above. Since

$$\lim_{n\to\infty}\left[1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log(n+1)\right]$$

exists, (2) follows at once.

From (2) we see that $\dfrac{A}{n^\mu} < a_n < \dfrac{B}{n^\mu}$, ...(3)

where A, B are positive constants.

Gauss's test (viz. that $\mu > 1$ is necessary and sufficient for the convergence of Σa_n) is an immediate deduction from (3). Bromwich's result is a further deduction.

If Σa_n is convergent, $\mu > 1$. Hence $na_n < \dfrac{B}{n^{\mu-1}} < \epsilon$ for all sufficiently large n. Therefore $\lim\limits_{n\to\infty} na_n = 0$.

If $\lim\limits_{n\to\infty} na_n = 0$, $\dfrac{A}{n^{\mu-1}} < na_n$, and therefore $\lim\limits_{n\to\infty} \dfrac{A}{n^{\mu-1}} = 0$.

This implies that $\mu > 1$, which in its turn implies the convergence of Σa_n.

It is obvious that our conclusion remains valid when (1) is replaced by

$$\log \frac{a_n}{a_{n+1}} = \frac{\mu}{n} + O\left(\frac{1}{n^\lambda}\right) \quad (\text{where } \lambda > 1). \quad \ldots\ldots\ldots\ldots\ldots(1')$$

II. When (1) or (1′) is fulfilled, the condition $\mu > 0$ is necessary and sufficient for the convergence of $\Sigma(-1)^{n-1}a_n$ (Bromwich, *loc. cit.*, § 19, p. 56).

To prove this we observe that when $\mu > 0$, (1) or (1′) ensures that $a_n > a_{n+1}$ for all sufficiently large n. Also, from (3), $\lim\limits_{n\to\infty} a_n = 0$. Hence, the convergence of $\Sigma(-1)^{n-1}a_n$ follows from Leibnitz's test for an alternating series.

If $\mu \leqslant 0$, we see from (3) that a_n cannot tend to zero as n tends to infinity. Hence $\Sigma(-1)^{n-1}a_n$ is non-convergent in this case.

It can now be proved, precisely as in I, that $\lim_{n\to\infty} a_n = 0$ is sufficient for the convergence of $\Sigma(-1)^{n-1}a_n$ when (1) or (1′) is true.

C. T. RAJAGOPAL.

1240. *The differential operator* $\left(x\frac{d}{dx}\right)^{-1}$.

The apparent discrepancy mentioned by Dr. Fox in his note, No. 1203 (July issue), as having been raised by one of his students, is a result of confusion between the operators $\left(x\frac{d}{dx}\right)^{-1}$ and $\frac{1}{x}\cdot\frac{1}{\frac{d}{dx}}$.

If P, Q are two operators, QP their product and $(QP)^{-1}$ its inverse, then

$$(QP)^{-1} = P^{-1}Q^{-1} \text{ and } \textit{not } Q^{-1}P^{-1}.$$

Thus, writing D for d/dx,

$$(xD)^{-1}f(x) = D^{-1}x^{-1}f(x) = \frac{1}{D}\cdot\frac{f(x)}{x} dx \int\frac{f(x)}{x}\,dx\,;$$

$$(xD)^{-2}f(x) = (xD)^{-1}\left[\frac{1}{D}\cdot\frac{f(x)}{x}\right] = \frac{1}{D}\cdot\frac{1}{x}\left[\frac{1}{D}\cdot\frac{f(x)}{x}\right]$$

$$= \int\left[\frac{1}{x}\int\frac{f(x)}{x}\,dx\right]dx\,;\ \text{and so on.}$$

In particular, $(xD)^{-1}x^{\alpha} = \alpha^{-1}x^{\alpha}$, and generally $(xD)^{-m}x^{\alpha} = \alpha^{-m}x^{\alpha}$, provided $\alpha \neq 0$.

On the other hand, $\frac{1}{x}\cdot\frac{1}{D}$ is the inverse of $D.\,x$.

Now $$(Dx)x^{\alpha} = Dx^{\alpha+1} = (\alpha+1)x^{\alpha},$$

$$(Dx)^2x^{\alpha} = (\alpha+1)(Dx)x^{\alpha} = (\alpha+1)^2x^{\alpha},$$

and generally $$(Dx)^m x^{\alpha} = (\alpha+1)^m x^{\alpha}.$$

Also $$(Dx)^{-1}x^{\alpha} = \frac{1}{x}\cdot\frac{1}{D}\cdot x^{\alpha} = \frac{1}{x}\cdot\frac{x^{\alpha+1}}{\alpha+1} = (\alpha+1)^{-1}x^{\alpha},$$

$$(Dx)^{-2}x^{\alpha} = (\alpha+1)^{-1}(Dx)^{-1}x^{\alpha} = (\alpha+1)^{-2}x^{\alpha},$$

and generally $(Dx)^{-m}x^{\alpha} = (\alpha+1)^{-m}x^2$, provided $\alpha \neq -1$.

Thus, summing up, we have

$(xD)^m x^{\alpha} = \alpha^m x^{\alpha}$,	$(Dx)^m x^{\alpha} = (\alpha+1)^m x^{\alpha}$,
$(xD)^{-m}x^{\alpha} = \alpha^{-m}x^{\alpha}$ $(\alpha \neq 0)$,	$(Dx)^{-m}x^{\alpha} = (\alpha+1)^{-m}x^{\alpha}$ $(\alpha \neq -1)$,

and the discrepancy vanishes.

The exceptional cases, when $\alpha = 0$ or $\alpha = -1$ respectively, are worth noting. They are:

$(xD)^m.\,1 = 0$,	$(Dx)^m.\,x^{-1} = 0$,
$(xD)^{-m}.\,1 = \frac{1}{m!}(\log x)^m$,	$(Dx)^{-m}.\,x^{-1} = \frac{1}{m!}\frac{(\log x)^m}{x}$.

H. A. HAYDEN.

1241. *The solution of a cubic equation.*

The method described below contains nothing new to the theory of cubic equations, but it does combine two methods, which are taught as though they were separate and unrelated, into one. It is quite possible that the method has been used previously, but at least it is not mentioned in the standard textbooks on the subject, so the repetition of the method, if this happens to be one, will at least bring it to general notice. The method is especially useful in obtaining the real root of a cubic.

Any cubic can be reduced to the form :

$$y^3 - 3Ay - B = 0. \quad \text{.....(i)}$$

The substitution employed is

$$y = b\{f(x) + a/f(x)\}, \quad \text{.....(ii)}$$

whence, $$y^3 = b^3\{[f(x)]^3 + a^3/[f(x)]^3\} + 3ab^2y,$$

or $$y^3 - 3ab^2y - b^3\{[f(x)]^3 + a^3/[f(x)]^3\} = 0. \quad \text{.....(iii)}$$

Putting $b=1$, and comparing coefficients of equations (i) and (iii), we have

$$1 = \frac{A}{a} = \frac{B}{\{[f(x)]^3 + a^3/[f(x)]^3\}},$$

i.e. $a=A$, and

$$[f(x)]^6 - B[f(x)]^3 + A^3 = 0, \quad \text{.....(iv)}$$

which is a quadratic equation for $[f(x)]^3$.

If this quadratic has real roots the cube root of the solution gives $f(x)$, and y, the solution of the original cubic equation is given as

$$y = \{f(x) + a/f(x)\}.$$

The necessary condition for real roots is, of course, $B^2 \geqslant 4A^3$. If this condition is satisfied by the coefficients in the initial equation (i), any function of x will serve the purpose of $f(x)$, and in particular x itself is satisfactory. Thus if inspection of the original equation shows $B^2 \geqslant 4A^3$ the substitution for a solution is

$$y = \{x + a/x\}.$$

This is the form of the substitution used by Cardan.

The solution when $B^2 < 4A^3$ depends quite naturally on a different value, and another property of $f(x)$. Before proceeding, however, it is appropriate to point out here that if A be negative B^2 is always greater than $4A^3$, so we need not now consider any case other than that in which A is positive.

Using the substitution

$$y = b\{f(x) + 1/f(x)\},$$

i.e. the substitution (ii) with $a=1$, we have equations similar to those obtained before, namely

$$1 = \frac{A}{b^2} = \frac{B}{b^3\{[f(x)]^3 + 1/[f(x)]^3\}}.$$

It is fairly obvious that the solution of the quadratic has to be dispensed with in order to avoid involving complicated imaginary quantities. A little consideration leads us to ask for what functions $f(x)$ does,

$$[f(x)]^3=f(3x).$$

If these functions exist, it is at once plain that

$$\begin{aligned}[f(x)]^3+1/[f(x)]^3&=f(3x)+1/f(3x)\\&=L(3x),\end{aligned}$$

where $$L(x)=f(x)+1/f(x).$$

If $L(x)$ has already been tabulated for all real x, then $L(3x)$ would give us, from these tables, the value or values of $3x$, and therefore of x. $L(x)$ is then read off directly from the tables.

We have, therefore, to find a function or functions $f(x)$ which obey the relation

$$[f(x)]^3=f(3x).$$

One such function is

$$f(x)=c^{dx}.$$

A difficulty that arises at this point is the fact that the function

$$L(x)=c^{dx}+1/c^{dx}$$

is not usually to be found in tabulated form except perhaps for $c=10$, and of course for $c=e$. We are therefore reduced to using either of the functions

$$f(x)=e^x \quad \text{or} \quad f(x)=e^{ix},$$

since the values of $L(x)$ are then known for all real x, namely

$$2\cosh x \text{ and } 2\cos x.$$

We have thus reduced the method of solution when $B^2<4A^3$ to the usual method arising from the equalities,

$$\cos 3\theta=4\cos^3\theta-3\cos\theta$$

and $$\cosh 3\theta=4\cosh^3\theta-3\cosh\theta.$$

We may generalise still further by seeking to find the functions $f(x)$ for which $[f(x)]^3=f(nx)$. By taking $n=0$ and 3 we see that both the cases $B^2\geqslant 4A^3$ and $B^2<4A^3$ are included under one head.

The link between the two common methods of solution of a cubic equation has been pointed out, and the most general fundamental substitution to supply this link is $y=b\{f(x)+a(f(x)\}$.

V. R. Coles.

REVIEWS.

Mathematics for the Million. By L. HOGBEN. Pp. 647. 12s. 6d. 1936. (George Allen & Unwin)

This is a remarkable book. It is original, it contains a great deal of mathematics, and most of this is set out in such a way as to be highly interesting. It is a book which at any rate all teachers of mathematics should take very seriously for it will obviously be of great use on the shelves of the classroom available for lending to pupils of certain types.

Whether it will be equally successful as a self-educator is another question. The non-mathematical (or failed-mathematical) reader is certain to find an increased interest in the subject if he follows the author's instructions to "read the whole book through once quickly"—which will for this type of reader imply skipping very many pages containing meaningless symbols—"to get a bird's-eye view of the social interconnections of mathematics".

It is in the "reading it for the second time" and "working on the detailed contents" that it seems likely there may be many failed self-educators.

The author "wrote this book in hospital for his own fun", but his real excuse for writing it and a general idea of its character may be given in two quotations from the first chapter.

"As mathematics has been taught and expounded in schools no effort is made to show its social history, its significance in our own social lives, the immense dependence of civilized mankind upon it. Neither as children nor as adults are we told how the knowledge of this grammar (the grammar of size and order) has been used again and again throughout history to assist in the liberation of mankind from superstition. We are not shown how it may be used by us to defend the liberties of the people. Let us see why this is so."

"This book will narrate how the grammar of measurement and counting has evolved under the pressure of man's changing social achievements, how in successive stages it has been held in check by the barriers of custom, how it has been used in charting a universe which can be commanded when its laws are obeyed, but can never be propitiated by ceremonial and sacrifice.

As the outline of the story develops, one difficulty which many people experience will become less formidable. The expert in mathematics is essentially a technician. So his chief concern in teaching is to make other technicians. Mathematical books are largely packed with exercises which are designed to give proficiency in workmanship. This makes us discouraged, because of the immense territory which we have to traverse before we can get insight into the kind of mathematics which is used in modern science and social statistics. The fact is that modern mathematics does not borrow so very much from antiquity. To be sure, every useful development in mathematics rests on the historical foundation of some earlier branch. At the same time every new branch liquidates the usefulness of clumsier tools which preceded it. Although algebra, trigonometry, the use of graphs, the calculus, all depend on the rules of Greek geometry, scarcely more than a dozen from the two hundred propositions of Euclid's elements are essential to help us in understanding how to use them."

These two quotations will perhaps suggest, as is certainly the case, that this first chapter entitled "MATHEMATICS, THE MIRROR OF CIVILIZATION" is an admirable introductory chapter eminently calculated to stimulate interest and encourage the reader to proceed.

The remaining chapters, with their very picturesque titles, will now be considered in turn. In each case remarks about the contents will be followed by quotations, chosen so as to indicate the variety of the author's comments on life in general and mathematics in particular.

The contents of Chapter II are indicated by its title, "FIRST STEPS IN MEASUREMENT OR MATHEMATICS IN PREHISTORY". It is of moderate interest; there are some attractive pictures in it. The examples are a rather dull set, mainly of the "draw a triangle and measure its angles" type. The interesting point is made that "primitive priestcraft ... owed its existence to the economic necessity of recording the passage of time and ... laid the first foundations of an organized body of scientific knowledge".

"Babylonian arithmetic just fell short of being an instrument of computation as efficient as our own. The principle of position was used consistently on a sexagesimal scale of 60. ... it employed a zero, which was intercalated to represent a gap in the sexagesimal series, as we might distinguish between 33 and 303. However, it rarely appears to have been used in the terminal position, as when we distinguish 33 and 330. This step seems to be all that was lacking to make the notation amenable to the Arabic devices we ourselves use."

CHAPTER III, "THE GRAMMAR OF SIZE, ORDER AND NUMBER OR TRANSLATING NUMBER LANGUAGE", is a very interesting one with many curious comparisons between mathematics and language. A striking point is the rendering of $=$ as "to get".

"The complete translation of the mathematical sentence $l \times b = A$ would read thus: the length must be multiplied by the breadth to get the area." [It is a bit awkward, by the way, if it is $A = l \times b$.]

"Mathematicians themselves do not call marks like $\times$ and $-$ verbs, they call them operators, just as Americans call workmen operatives. This means that they are not simply there for ornament like dukes or beefeaters. They do real work."

"Before Ogden and Jespersen we had no real grammar of English. What we have been taught as English grammar is a pretence that every rule in Latin must have some equivalent rule in English."

The exercises to Chapter III include first examples in Algebra. It is a pity that there are no answers. It is hard to imagine that anyone teaching himself through this book could be sure that he had the simplified form of

$$\frac{(3ab^2)^3}{9a^2b^5}$$

correct or the value of x in terms of a and b, when $a(a-x) = 2ab - b(x+b)$.

In checking with numbers it would be easy to repeat a mistake made with the letters.

CHAPTER IV, "EUCLID WITHOUT TEARS OR WHAT YOU CAN DO WITH GEOMETRY", is perhaps the least successful chapter in the book. It may be impossible for two lines to be exactly equal but it is no easier for them to be exactly equivalent, and to keep on replacing "equal" by "equivalent" is not an abbreviation but merely an irritation.

The choice of the various "rules" and "demonstrations" does not strike one as specially happy, but there are some nice pictures among the applications.

The exercises at the end of the chapter will be found hard.

The author keeps on alluding to the Asses' Bridge, but does not seem to know what diagram (with at anyrate a likeness to something in the "real world") was thus named. This somewhat spoils his various jokes about it.

"When the Phoenician Thales discovered how to inscribe a right-angled triangle with its corners in a semicircle, he sacrificed an ox to the gods. It was a bad business for the ox. In the end it was a bad business for the gods."

"At the very point when it had created a new instrument for man's conquest of the world in which he lives, geometry degenerated into a mere toy. Only when Greek civilization was destroyed were the fruits of Greek geometry harvested."

CHAPTER V, "FROM CRISIS TO CROSS-WORD PUZZLES OR THE BEGINNINGS OF ARITHMETIC", is a most discursive but most interesting chapter. Magic squares, Gamatria, figurate numbers, with nice diagrams of pentagonal, hexagonal, stellate numbers, and so on; combinations, factorials, probability, interspersed with remarks on the magical properties of numbers and their effects on history, are nicely blended.

"To say that mathematics is the grammar of science implies that science is concerned with nothing more than enunciation and measurement. The plain fact is that the first task of science is to recognize what different sorts of things there are in the world." "Nothing but confusion has resulted, and can result, when mathematics is used before we are quite clear about the sort of things with which we are dealing and what sort of measurements it is useful to make."

"In the field of psychology the amount of arithmetic devoted to intelligence-testing is out of all proportion to its substantial basis of enduring fact. We have not fully replaced the sacramental by the instrumental attitude to the use of number."

CHAPTER VI, "THE SIZE OF THE WORLD OR WHAT WE CAN DO WITH TRIGONOMETRY", is probably the most interesting in the whole book. With a very little rewriting it would be worth publishing separately with some such title as "the achievements of the great Alexandrians". The sympathy of the author with these is perhaps shown by the "we" in the title as contrasted with the "you" in the title of Chapter IV.

"What we can do with Trigonometry" includes measuring the size of the Earth, working out Trigonometrical tables and formulae, finding the distances of the Sun and Moon, surveying the Earth, finding the value of π, using radians, the law of signs in Algebra, and the Alexandrian multiplication table (with letters), finding square roots and looking at conics.

In spite of the interest of the material, the self-educator will have a hard time here, with lots of arithmetic in calculating sines; the proofs of the cosine formula, half-angle and $A+B$ formulae to master, and a fearsome Japanese method of getting π with series of square roots

$$\frac{4}{n^2}(n+\sqrt{n^2-1^2}+\sqrt{n^2-2^2}+\sqrt{n^2-3^2}+\ldots)$$

to be dealt with in the text, while in the examples he is invited to deduce "the factors of $\sin A+\sin B$, etc., and hence show how the half-angle formulae can be built up from" the addition formulae. It is about here that the man who can give a few hints is likely to become a necessity.

"The history of science never confirms the popular belief that discoveries are made singly by the isolated genius. They are invariably made about the same time by a number of different people. This is a commonplace of scientific history, though historians of science rarely disclose the simple explanation. Discoveries repeat themselves because intellectual work proceeds along the lines laid down by the social culture which the individual discoverer inherits."

"Another way of measuring the angle might be called the wheelwright's way, which we may call the unit of the machine age. The machine unit which is called the RADIAN . . . ".

"In putting back measurement into geometry, and so bringing geometry back to earth, Alexandrian materialism attained a new vision of the grandeur of the heavens. The vastness of the prospect engulfed the 'heavenly race of the gods' from whom Plato derived the ancestry of philosophers."

"Archimedes did not succeed in reforming the number script of his contemporaries nor in making tables of logarithms by which any multiplication can be carried through rapidly. Such a change would have meant uprooting the social culture of his time. People were still using the old notation for *low*

numbers. His brilliant failure shows that we cannot afford to let the mass of mankind be uneducated however much entertainment the eugenist may get from reflecting upon his superiority to the rest of his fellows."

"Men who were not advanced mathematicians had invented a symbol (0) for nothing. There is no more fitting epitaph for the decay of Alexandrian science . . . than . . .

the stars are setting and the caravan
Starts for the Dawn of Nothing, oh make haste."

CHAPTER VII, "THE DAWN OF NOTHING OR HOW ALGEBRA BEGAN". In spite of its poetical title this chapter is less successful than others. In it are discussed separately "the need for simple rules of calculation or algorithm, the solution of practical problems involving numbers in the solution of equations and finally the renewed interest in series."

Turning to the second of these we find that "solving problems is not a special gift but merely the art of applying fixed rules of grammar" and have the novel rule. Translate each separate item of information into the form "By or with something do something to get something", of which an instance from one of a series of problems worked out in detail is, "By 5 multiply my son's age to get my father's age".

The work on series, which includes the Binomial Theorem (positive integral index), and touches on finite difference notation will not be found easy by the self-educator. "Choke off" is a nice phrase for "converge".

"The algebra (of the Alexandrines) was hopelessly complicated by the fact that they used the same symbols for words and proper numbers. So they could not confine the use of letters to abstract numbers."

"'Physician and algebraist' was a term still used in Spain until quite recently, just as surgeon and barber went together in the Middle Ages."

The examples at the end of this chapter look like a not very exciting selection from any school algebra and are likely to be found difficult.

CHAPTER VIII, "THE WORLD ENCOMPASSED OR SPHERICAL TRIANGLES". This is admittedly a difficult chapter for which the author almost suggests skimming, as "later chapters do not depend on it". However, excellent diagrams and careful explanation have done much to make it reasonably possible.

"(For the sundial). All that remains is to graduate the scale in hours. You can get some amusing practice in Trigonometry by doing this. If at the same time you remember that the theory was worked out by the coloured conquerors of Spain when the people of Britain and Germany were barbarians, living in mud huts, ruled by ignorant priests and robber barons, it will also clear your mind of the sort of cant chanted by the white settlers of Kenya, General Hertzog, the apostles of selective immigration restrictions, and the pundits of German national socialism."

CHAPTER IX, "THE REFORMATION GEOMETRY OR WHAT ARE GRAPHS?" This is a chapter the material of which is of extreme interest though the explanations appear somewhat clumsy in places.

"At the close of the fifteenth century a new era began. The manufacture of wheel-driven clocks, the introduction of artillery into warfare, the preparation of maps and astronomical tables for navigation forced mathematics to grapple with *time* in a new social context with a new technical equipment."

To illustrate coordinates and the equation of a circle, the origin is taken at Long. 0°, Lat. 0°, and a circle whose radius is 50° drawn. This passes close to Nicaea.

"Nicaea is noticeable for several other reasons. A hundred years after the schools of Alexandrian science were destroyed by the monks, the city of Nicaea was chosen as a centre for celibate arithmetical researchers into the properties

of the number 3. The familiar results of this energetic erudition are much more difficult to comprehend than the table of natural logarithms, and since capital punishment for mistakes in using the Nicene creed has claimed a host of victims, we are doubly fortunate in being able to learn the mercantile arithmetic of the Reformation instead."

In this chapter we have equation of circle, straight line (complete with dy/dx as gradient), cannon-ball parabola, cubic curves, curves to illustrate Hooke's Law and the law of cooling, the ellipse as planet's path, extensions of trigonometrical ratios to large angles, wave curves, the curve for 2^x, and various algebraic curves. x, y, z coordinates are introduced, also polar coordinates, and delightful diagrams of a fish and some skulls drawn first with ordinary and then with curvilinear coordinates.

The exercises, which seem more probable for the self-educator than for several chapters back, cover, like the text, a comprehensive course of graphs and their applications.

"In Euclid's timeless geometry an angle is a fixed and ready-made quantity. In Reformation geometry it has a history. It is generated by the movement of a line about the x-axis."

"In contrast with the mathematics of the Greeks, the mathematics of the Reformation was dynamic. It took time into account. In contrast with the requirements of a mathematics which would be necessary to deal with the whole range of modern knowledge Reformation mathematics is *non-historical*. . . . We are beginning to see that a mathematics which allows us to move is not enough. We need a mathematics which can concern itself with whence we came and whither we are going."

CHAPTER X, "THE COLLECTIVIZATION OF ARITHMETIC OR HOW LOGARITHMS WERE DISCOVERED." This is one of the less successful chapters. However, it covers much ground. The discussion of logarithms leads to series for making tables, in which the general binomial series, the exponential series, de Moivre's theorem, and the series for cosine and sine are reached. A short section on the addition of vectors ends the chapter.

The methods strike one as clumsy. No use is made of the beginner's method of reading logarithms from a graph nor of the fact that a change of scale converts any exponential curve into any other. The elementary method of calculating logarithms set out in the M.A. report on Algebra is not mentioned though it involves far less labour than those that are given. The method of deriving the exponential series from the binomial series might have been chosen as that involving the maximum amount of arithmetic.

It might also be mentioned that the author is so keen on dividing up the credit for logarithms that he leaves precious little for Napier and that Newton is never mentioned in connection with the binomial theorem (of Omar Khayyàm).

The exercises at the end of the chapter are a sensible and attractive set, and involve far less arithmetic than is implied in following through the text of the chapter.

"By comparison with such calculations as had been undertaken by the Alexandrian mathematicians, the tasks which arose from the expansion of trade and improved technique in navigation during the fifteenth century made exorbitant demands upon the *Rechenmeister* and compelled the search for more compact and less laborious algorithms than those which we learn in our childhood from the Arab schoolmasters of Western civilization."

"The two rules which we have given for fractional and negative powers were first stated by Oresmus in a book called *Algorismus Proportionum*, published about A.D. 1350. It took the human race a thousand years to

bridge the gulf between the rule of Archimedes and the next stage in the evolution of the logarithm table. You need not be discouraged if it takes you a few hours or days to get accustomed to the use of a fractional or negative power."

"Having got so far, it seems a great pity that Luther's convert (Stifel) spent so much time in arithmetical calculations to prove that Pope Leo X was the Beast of the Apocalypse, instead of undertaking the socially useful task of compiling tables like those of Bürgi or Briggs."

"A seventeenth-century mathematician deduced irrefutable evidence for the creation of the world out of nothing from the behaviour of this series

$$(1-1+1-1+\ldots).$$"

"Do not bother at present about the *meaning* of this

$$[(\cos\alpha+i\sin\alpha)^n=\cos n\alpha+i\sin n\alpha].$$

The meaning of a rule in mathematics is either a statement about its consistency with other rules, or a statement about how it can be used. First satisfy yourself that it is consistent with what you know already . . . ."

CHAPTER XI, "THE ARITHMETIC OF GROWTH AND SHAPE OR WHAT THE CALCULUS IS ABOUT." This is a chapter of moderate interest and a good deal of difficulty. It does not make the subject nearly as simple as it seems to the reader of the well-known "Calculus made easy"—perhaps a completely unfair comparison as in this chapter the progress is much more rapid. Beginning with the familiar kinematical illustrations of rate of change, "this chapter chiefly uses the infinitesimal calculus to solve numerical problems which do not depend on understanding mechanics". Algebraic, exponential and trigonometrical functions —of course only fairly simple ones—are differentiated and integrated; the logarithmic series and Gregory's series obtained; and applications made to find the volume of a sphere, the energy of a stretched spring and the efficiency of an internal combustion engine. The general effect is more scrappy than in some of the other chapters, but the examples at the end are fairly reasonable.

"Since there are still ignorant people who produce average measurements of the brain capacity of natives in backward regions as a reason for withholding educational opportunities from them, it is worth while remembering that the cranial capacity of Leibniz, like that of Anatole France, was lower than the average figure for any aboriginal population."

"The explosion of the cannon and the ticking of the clock were the tocsin of a new phase in the organization of man's social life."

"To know how efficient a machine is we need to be able to calculate how much potential energy is wasted. The solution of this problem gave the methods of the integral calculus much greater practical importance than its inventors could have realized."

"Nothing would do more to promote a rapid advance in human understanding than an annual conference between schoolboys, schoolmasters and elderly scientists. The scientists should be compelled to attend under pain of losing their pension rights."

CHAPTER XII, "STATISTICS OR THE ARITHMETIC OF HUMAN WELFARE". This very interesting chapter deals with probability and its connection with the binomial expansion; statistical hypotheses in natural science, especially in connection with Mendel's law; the theory of errors and the normal curve of error; statistical theory in social science including Spearman's rank-coefficient; variance as a measure of dispersion; life tables and their construction from death statistics. In an appendix the median and the mode are defined and the correlation coefficient discussed. The examples at the end are few but suitable.

A great part of the chapter discusses the cautions necessary in using the mathematical results. No attempt is made to prove the equation of the normal curve, but the value of the " rank " coefficient is proved.

On the whole the explanations in this chapter are very clear and readable.

" Whether the speculator succeeds depends on how long he can go on putting up the capital required. Successful speculation therefore depends on the principle ' To him that hath shall be given, and from him that hath not shall be taken away even that which he hath '."

" The doctrine that all scientific laws are statistical is a statistician's view of what the scientist ought to do. It is not a true description of what the scientific worker actually does. . . . The scientist is always seeking to get a better approximation."

" While recognizing its merit for what it is worth, we must guard against the error that a difference the statistician calls significant has any real significance from the social standpoint. In the last example, the socially important feature of the statistics cannot be represented by an average ; so a difference between two such averages may be statistically significant and socially insignificant."

" EPILOGUE ON SCIENCE OR MATHEMATICS AND THE REAL WORLD " gives in an interesting way some cautions to be observed in translating mathematical formulae back into English. Boyle's law in the form $v=R/p\,.\,t$ can be used as a recipe for action if translated " raise the temperature of a gas at a fixed pressure so much to increase its volume by so much ". It becomes meaningless if translated " raise the volume of a gas at a fixed pressure so much to increase its temperature by so much ".

" A mathematical statement can only be used to prescribe a correct course of action if it is supplemented by *additional* knowledge derived from experience of how a process actually occurs in the real world."

Further, " the great precision with which the rules of mathematical discourse are stated does not imply that a description of nature is necessarily more exact because the language used to describe it is mathematical ".

Again, extrapolation in graphs is to be avoided, " the extrapolations of astronomers and geologists do most harm to their own reputations. Since Kelvin's colossal blunder about the rate of cooling of the earth was exposed by Madame Curie's discovery of radium, specialists in natural science treat similar exploits as an amusing and permissible branch of fiction ", but " Professors of economics still believe that it is legitimate. . . ."

Scientific men use the word *law* in two ways. " One is for *observed regularities of nature.*" A striking example is given of the other. " When we say that *action and reaction are equal and opposite* is a scientific ' law ' we convey no information about newly observed regularities of nature or guidance for the correct conduct of anybody except a mathematician. What we really do is to state briefly in general terms the kind of symbolic conventions we intend to adopt in applying the kind of mathematics we intend to use when making measurements upon the sort of situation about which we are speaking. If we must use the word law for statements of this class, we should call them bye-laws, implying our right to repeal them."

Newton, having been regarded as a great scientist, will presumably be thought of as a great bye-law-giver, still his " system remains a good enough equipment of calculating tricks for communicating the real laws of Nature revealed by any observations which had been made till long after his death."

Nor must we be pessimistic as to future Newtons. " When the educational powers of the cinema have been adopted to visualizing the use of mathematical

symbols, anything which can be achieved by books and blackboards will seem trivial by comparison."

In conclusion, it may be asked how far the author succeeds in proving his main thesis that all valuable discoveries in mathematics are due to the demands of the social environment of the discoverers.

In endeavouring to prove this he certainly raises to the *n*th power the method of the Bellman (" what I tell you three times is true ") and does so with such a variety of language that his argument might almost be praised, as is the Scherzo of the Choral Symphony, " a miracle of repetition without monotony".

It must be admitted that he makes very effective play with the importance of the invention of the sign for zero, and of course he reduces the difficulty of his thesis very much by insisting as a matter of definition that, to be valuable, a discovery in mathematics must react on the real world and affect the social environment of the discoverer. This gets him into difficulties once or twice, as when Mendel's laws provide a use for Fibonacci series or the discovery of the mechanical equivalent of heat " gave the methods of the integral calculus much greater practical importance than its inventors could have realized ", but on the whole helps him to make a good case, and one which will probably be regarded as conclusive by his non-mathematical readers.

May he have as many of them as his title demands! If they don't all succeed as self-educators, perhaps the demand for private tuition will help the professional teacher; and anyhow he has been benefited by the production of a book which should provide him with many new ideas and dodges for teaching, much valuable historical information, and a good deal of amusement.

C. O. TUCKEY.

Back to Newton. A challenge to Einstein's Theory of Relativity. By G. DE BOTHEZAT. Pp. vii, 152. $2.50. 1936. (Stechert, New York)

The author's attitude to nineteenth and twentieth century mathematics and mathematical physics is rather like that of the recruit who said " They are all out of step except me! " After giving what he considers to be the rigorous definitions of variable infinity and absolute infinity, he concludes that " the whole theory of aggregates of Georg Cantor starts with a stupendous fallacy. ... Cantor's conceptions become untenable . . . they have only entangled the whole subject in fallacious, sophisticated and useless arguments ". Turning to geometry, we read that " the difficulties met for centuries in the analysis of Euclid's postulate on parallels were exclusively due to a lack of a sufficiently rigorous grasp of infinity. . . We can then conclude with perfect rigor: *Through a point taken outside of a line one can draw an infinity of parallel lines, but these lines all coincide among themselves for any finite distance counted from the considered point. Any two parallel lines make an infinitesimally small angle among themselves.* . . . This definition " (of infinity) " once achieved Euclid's postulate " (as restated above), " cannot be any more at will rejected, and the non-Euclidean geometries cannot any more be accepted as a generalization of Euclidean geometry ".

The next subject dealt with is that of Absolute Time, which is defined by the behaviour of ideal clocks. It is argued that the impossibility of constructing such clocks gives no more reason for rejecting the theory of absolute time than does the impossibility of constructing exactly the concepts mentioned in geometry for rejecting geometry. As for motion, " Motion is essentially relative. But as every configuration can involve only a finite kinetic energy—mostly oscillating between two extreme values—this determines unequivocally in every case a single frame of reference, which has the privilege that, in relation to same, the kinetic energy involved is directly and correctly estimated. We call *absolute motion*, motion in relation to such privileged frame

of reference". After a discussion of Newton's famous experiment of the rotating vessel filled with water, it is concluded that "it is easy to show *the complete fallacy of Einstein's Principle of equivalence*. . . . The big misunderstanding at the ground foundation of Einstein's theories of relativity is after all an entirely erroneous conception of relative motion".

It is rather surprising to be told that "not a single conclusion, *derived from Einstein's General Theory of Relativity, has ever been experimentally verified.* This is to-day fully admitted by the crushing majority of the leading physicists". References are given to E. Reichenbacher (*Proc. German Math. Soc.* xlii, p. 132, 1933) and C. Schaefer (*Einführung in die Theoretische Physik*, vol. iii, part i, 1932).

In the Epilogue, the author attributes modern errors to the degradation caused by the great war. But he knows that "there have been preserved in this World a few isles still inspired by competent dignity and enlightened decency" . . . and does not doubt that "these remains of generous altruism, profound cogniting erudition and well meaning will to achieve are numerous, self-conscious and determined enough to overcome the dark and perfidious forces engaged in the destruction of our culture and civilization". H. T. H. P.

A Course in Mathematics for Craftsmen, I. By S. L. Whitby. Pp. x, 222. 5s. 1936. (Pitman)

This book "is intended primarily for students preparing for the Handicraft Teachers' Examination of the City and Guilds of London Institute, but it is hoped that it will be of service to other Technical Students". We should have thought that most of it ought to have been covered in any ordinary school course. It consists of fifteen chapters, mainly on simple arithmetic: fractions, decimals, metric system, mensuration, etc., with three chapters on algebra up to simple simultaneous equations, and three on geometry, ending with Pythagoras' theorem. The algebra and geometry are interpolated into the arithmetic in what is doubtless a useful sequence in the early stages. The book is well produced and cheap for its size. F. B.

A Textbook of Trigonometry for colleges and engineering schools. By W. H. H. Cowles and J. E. Thompson. Pp. x, 373. 12s. 6d. 1936. (Chapman and Hall)

This book should prove remarkably useful to any student wishing to acquire a knowledge of the elements of the subject, whether from the purely theoretical standpoint, or with a view to its applications to more practical problems.

No previous acquaintance with the subject is necessary, nothing but the merest elements of Euclid and algebra being assumed. The chapters follow on from one another in logical sequence, starting with the ordinary definitions of the trigonometrical ratios and finally leading up to their expansions in series, De Moivre's Theorem, and the allied results.

For the convenience of the practical student, two chapters, 5 and 11, are devoted to some of the more interesting and useful applications of the bookwork to problems in geodesy, astronomy and surveying. The engineer should also derive considerable benefit from the frequent references to the use of the slide rule, in particular from Chapter 16, in which is given a clear and concise account of its everyday uses in arithmetical and trigonometrical computation.

An excellent selection of examples is available at the end of each chapter.

J. H. P.

Électricité. By J.-B. Tourriol. Pp. 320. 65 fr. 1937. (Gauthier-Villars)

This textbook of Electricity and Magnetism forms part of a college course in physics for those specialising in mathematics. (The author had previously

published volumes on Geometrical Optics and on Heat.) The standard of difficulty is approximately that of a "pass" or "general honours" degree course; but only the main parts of the subject are treated.

The first 109 pages are devoted to electrostatics; then follow sixty-six pages on magnetism; and the remainder of the book consists of sections on electrodynamics and electro-magnetism. There are eighteen pages of examination questions. The book is mainly theoretical, experimental matters being clearly described, but without full practical details.

The section on electrostatics includes a discussion of the capacities of systems of conductors, the energy of charged condensers, and the work done when the capacity is changed. In magnetism, the (I, H) curve for iron, and hysteresis are treated briefly; and the theory is given of magnetic shells and of the refraction of lines of induction at a surface. The treatment of current electricity ends with a chapter on the action of a magnetic field on a current, and a chapter on the application of the laws to electrical measurement. Alternating currents are not discussed, though the theory of the ballistic galvanometer is given; and none of the modern discoveries in atomic physics are mentioned.

The fundamental parts of the subject are developed with great logical precision. Incidentally, it is pleasing to find the dimensions of quantities measured on the electrostatic and electromagnetic systems given correctly; many English textbooks give by mistake the dimensions that would obtain *if* we used arbitrary standards of permeability and dielectric constant (preserved at some National Laboratory).

There is one criticism that I must make. Gauss' theorem is given in the form

$$\Phi=\frac{4\pi}{k}(q_1+q_2+q_3+\ldots),$$

which is only applicable when k is constant over all the surface considered.

W. N. Bond.

Elementary Analytical Conics. By J. H. Shackleton Bailey. Pp. 375. 7s. 6d. 1936. (Oxford)

This book is intended primarily for higher certificate candidates and it includes many worked examples "designed to familiarise students with the processes usually employed in solving problems set at examinations".

A book ought to have a higher aim than that, such for example as the teaching of geometry.

The definition of a tangent is given in an archaic form. No reasonable explanation is offered of what is meant by "imaginary points" and yet the polar is defined as the line joining the points of contact of tangents (even for an internal point). To a person ignorant of mathematical ideas, says the author, it might seem ludicrous to speak of tangents to a circle from an internal point. It might indeed. And what does the mathematician think about an internal point of a complex circle? Does he even define it?

Determinants occur in the book but are not used as freely as might be desired. There is perhaps more than usual about parameters; the limitation to conics makes it difficult to show clearly what is their place in geometry. It is wrongly stated that $(a\,\mathrm{ch}\,\phi, b\,\mathrm{sh}\,\phi)$ can be taken as the coordinates of any point on the hyperbola. The slurring of a detail of this kind confirms the impression that the author is not making a serious attempt to teach geometry. Another example is the explanation of the sign of ax_1+by_1+c for which the reviewer would unhesitatingly assign a zero mark to a beginner.

Envelope equations are defined as the condition of tangency for

$$lx+my+n=0.$$

The dual aspect of a curve as a locus or envelope is not made clear.

The book is well printed and well arranged. It gives about fifty pages each to the favourite topics of higher certificate pass papers in geometry. There is a danger that it may appeal to those teachers who are chiefly interested in the passing of examinations. The blame for this will lie partly with those who set the papers for higher certificates. Or perhaps it is a change of regulations that is required, so that candidates have to qualify in coordinate geometry instead of in conic sections. A. R.

1. Magic squares of $(2n+1)^2$ cells. By M. J. VAN DRIEL. Pp. 90. 10s. 6d. 1936. (Rider)

2. Easy methods for the construction of magic squares. By MAJOR J. C. BURNETT. Pp. 77. 2s. 6d. 1936. (Rider)

I know as much about magic squares as most people. Indeed most people know less than I. But I am not a specialist. And only a specialist can review these books adequately.

The first is not a translation of two books by the same author recently noticed in the *Gazette*. It claims to be new research in an old field. It is a book for the serious student and it will require serious study to understand the text; for the translation does not achieve lucidity.

It opens with Euler's method with several examples, using the notation

1^1	2^5	3^4	4^3	5^2
2^2	3^1	4^5	5^4	1^3
3^3	4^2	5^1	1^5	2^4
4^4	5^3	1^2	2^1	3^5
5^5	1^4	2^3	3^2	4^1

The numbers 1, 2, 3, 4, 5 are placed in each row so that no number occurs twice in the same row or column and the indices are placed to fill the same requirements. Each number a^b is now replaced by $5(a-1)+b$. The square obtained is obviously magic in rows and columns, but not necessarily in diagonals. This method with the possible variations of arrangements is exhaustively treated. There follow chapters on a uniform step method, bordered magic squares, magic squares containing a rectangle which can change position, and so on.

The second book uses simple equations connecting complementary differences and develops a systematic method of constructing associated squares, that is, squares in which two numbers symmetrically situated with regard to the centre add up to the square of the order. The same method is applied to bordered magic squares and to squares formed by using magic rectangles. The last section deals with pandiagonals or Nasiks and the transformation of one type into the other.

I hope that this summary is not unfair to the books and will help enthusiasts to see what is their utility and interest. F. C. B.

Bibliotheca Chemico-Mathematica. Second supplement. I, II. Compiled and annotated by H. Z(EITLINGER). Pp. xi, 1-840, 841-1396. 21s. 1937. (Henry Sotheran)

The new supplement to *Bibliotheca Chemico-Mathematica* contains about 23,000 entries, of which about one-sixth concern mathematics, including a list of portraits and autographs. The remainder of the work is devoted to the natural sciences and particularly to their technical applications.

These volumes are, like their predecessors, indispensable to the collector, the librarian and the historian. As far as the mathematician is concerned, the comments are still an instruction and a delight, and tempt us to re-christen the book as "The Best Short Stories of 1937". T. A. A. B.

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